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# UNIFIED THEORY FOR THE BENDING AND BUCKLING OF SANDWICH SHELLS APPLICATION TO AXIALLY COMPRESSED, CIRCULAR CYLINDRICAL SHELLS

By

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July 1971



EUSTIS DIRECTORATE
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The report presents the results of research conducted in the analysis of axially compressed, circular cylindrical shells by means of a variational development of the shell equations and boundary conditions for bending and buckling.

The report has been reviewed by the Eustis Directorate, U. S. Army Air Mobility Research and Development Laboratory, and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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| Axial Compression | | | | | | |
| General Instability | | | | | | |
| Face Wrinkling Instability | | | | | | |

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FORT EUSTIS, VIRGINIA

## SUMMARY

A variational development of the governing equations and boundary conditions for the bending and buckling of sandwich curved plates and shells is presented. The theory is applicable to sandwich construction of the honeycomb type and may be used to describe bending, buckling, and post-buckling behavior. With the coupled effects of core transverse shear and normal strain included, the theory reflects the presence of 8 degrees of freedom and 2 constraining conditions. Nine boundary conditions per curved plate or shell edge can be specified; apart from the four of the usual type for each of the sandwich facing elements, in terms of either a generalized force or displacement, a condition for either the shear loading or the average displacement of the core is obtained. The governing equations, specialized for small-displacement buckling, are applied to the case of axial compression of circular cylindrical shells. The unified theory analysis reflects three modes of buckling: classical general instability and both symmetric and antisymmetric wrinkling. Comparison of the results with published experimental data for honeycomb-core sandwich shells of practical dimensions yields very strong evidence to suggest that the axial compression buckling load and failure mode of such shells, unlike homogeneous, isotropic shells, are given in good approximation by a linear theory.

## FOREWORD

The work reported herein constitutes a portion of a continuing effort being undertaken at Stanford University for the U.S. Army Aviation Materiel Laboratories* under Contract DA 44-177-AMC-115(T) (Task 1F162204A17002) to establish accurate theoretical prediction capability for the static and dynamic behavior of aircraft structural components using both conventional and unconventional materials. A predecessor contract supported investigations which led, in part, to the results presented in References 2 and 4.

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## LIST OF SYMBOLS

| | |
|---|---|
| $A$ | operator matrix |
| $\|A\|$ | determinant of A |
| $\|\|A_{ij}\|\|$ | matrix of cofactors of entries of A |
| $\|\|A_{ij}\|\|'$ | adjoint of A (transpose of $\|\|A_{ij}\|\|$) |
| $A,B,C$ | polynomials of wavelength parameter $\beta$ |
| $C$ | extensional rigidity of plate = $Et/(1-\nu^2)$ |
| $D$ | flexural rigidity of plate = $Et^3/12(1-\nu^2)$ |
| $D_s$ | flexural rigidity of sandwich plate with identical membrane faces = $Et(c+t)^2/2(1-\nu^2)$ |
| $D^*$ | generalized flexural rigidity |
| $E$ | Young's modulus of a face material |
| $E_t$ | tangent modulus of a face material |
| $\bar{E},\bar{E}_t$ | modified moduli of a face material $\left(E/(1-\nu^2)\ ,\ E_t/(1-\nu^2)\right)$ |
| $E^*$ | generalized modulus of face material |
| $\bar{E}^*$ | generalized reduced modulus |
| $F$ | elastic modulus for transverse core strain |
| $F^*$ | generalized modulus for transverse core strain |
| $G_1,G_2$ | shear moduli for transverse shear of the core layer |
| $G^*$ | generalized shear modulus for the core |
| $I$ | unit matrix |
| $J_2$ | second invariant of stress deviator tensor |
| $L$ | linear operator; length of sandwich cylinder |

| | |
|---|---|
| $M_x, M_\varphi$ | components in the x and $\varphi$ directions, respectively, of the edge moment per unit length of face layer of sandwich cylinders |
| $M_s, M_n$ | components in the s and n directions, respectively, of the edge moment per unit length of face layer of a sandwich shell |
| $N_x, N_\varphi, N_\zeta$ | components in the x, $\varphi$, and $\zeta$ directions of the edge force per unit length of face layer of a sandwich cylinder |
| $N_s, N_n, N_z$ | components in the s, n, and z directions of the edge force per unit length of face layer of a sandwich shell |
| N | curvature dependent factor appearing in connection with Equation (36) $\left(= \zeta\epsilon^2\nu^2/(1-\nu^2)\right)$ |
| R | domain occupied by the core layer |
| $R, R_1, R_2$ | radii of the middle surface of the core layer, the outer face layer and the inner face layer, respectively, of a sandwich cylinder |
| S | surface domain occupied by the face layers |
| $\partial S$ | boundary curve of domain S |
| T | total potential functional (= U + V) |
| U | strain energy functional |
| V | potential functional of applied loads |
| U,V,W | amplitudes of displacement components u, v, w |
| a | diameter of inscribed circle of hexagonal core cell |
| c | thickness of the core layer |
| $d_s, d_n$ | displacement components in the s and n directions, respectively |
| $c_{ij}, d_{ij}$ | transformation symbols defined in Equation (90) |
| f | factor involving core orthotropy (Equation 60) |
| f,g | functions of integration (integrated form of Equation 97) |

| | |
|---|---|
| $k$ | load parameter $(= N_x c^2/\pi^2 D)$ |
| $k_1, k_2$ | load parameter for symmetric and antisymmetric deformations, respectively, of a circular sandwich cylinder |
| $m, n$ | number of axial and circumferential waves |
| $n$ | coordinate normal to shell boundary |
| $p$ | square of wave aspect ratio $(= \lambda_x^2/\lambda_\varphi^2)$ |
| $p_1, p_2$ | lateral surface loads on the two faces of a sandwich shell |
| $q_1, q_2, q_3$ | components of the interface traction in the $\xi_1$, $\xi_2$, and $\zeta$ coordinate directions, respectively |
| $q^*$ | edge shear load on the core of a sandwich shell |
| $\bar{r}$ | column vector |
| $s$ | curvilinear coordinate along boundary of shell |
| $t_1, t_2$ | thickness of the face layers |
| $\bar{u}, \bar{u}^*$ | generalized displacement vectors |
| $u, v, w$ | displacement components in the $\xi_1$, $\xi_2$, and $\zeta$ coordinate directions, respectively |
| $\bar{u}, \bar{v}, \tilde{w}$ | average displacement components $(u_1+u_2/2$ , etc.) |
| $\tilde{u}, \tilde{v}, \bar{w}$ | relative displacement components $(u_1-u_2/2$ , etc.) |
| $u^*, w^*$ | prebuckling displacement components in axial and radial directions of a sandwich cylinder |
| $x$ | axial coordinate of sandwich cylinder |
| $z$ | coordinate normal to middle surface of a face layer |
| $\alpha_1, \alpha_2$ | metric components of a surface for the 1 and 2 coordinate directions, respectively (square of length of line element: $\alpha_1^2 d\xi_1^2 + \alpha_2^2 d\xi_2^2 + d\zeta^2$ ) |
| $\alpha_s, \alpha_n$ | metric components for s and n coordinate directions |
| $\bar{\alpha}$ | value of $\alpha$ on reference surface |

| | |
|---|---|
| $\beta$ | wavelength parameter $(= c/\lambda)$ |
| $\delta$ | variational operator |
| $\epsilon$ | geometric parameter $(= t/c)$ |
| $\epsilon_{eff}$ | effective strain |
| $\epsilon_1, \epsilon_2, \gamma_{12}$ | plane strain components of a face layer |
| $\epsilon_3, \gamma_{13}, \gamma_{23}$ | antiplane strain components of the core layer |
| $\gamma_1, \gamma_2$ | value of $\gamma_{13}$ and $\gamma_{23}$ on the middle surface of the core |
| $\zeta$ | coordinate along normal to middle surface of the core; curvature parameter $\left(= \frac{12(1-\nu^2)}{\pi^4} \frac{c^4}{t^2 R^2}\right)$ |
| $\eta$ | efficiency parameter (test load/theoretical load) |
| $\theta$ | stiffness parameter $\left(= 6\pi^2 \frac{D}{c^3 \sqrt{G_1 G_2}}\right)$ |
| $\theta^*$ | reduced stiffness parameter $\left(= \frac{\bar{E}}{E} \frac{G_1}{G_1^*} \theta\right)$ |
| $\kappa_1, \kappa_2, \kappa_{12}$ | changes of curvature and twist for the 1 and 2 directions of a middle surface of a face layer |
| $\lambda$ | wavelength |
| $\lambda_x, \lambda_\varphi$ | wavelengths in axial and circumferential directions of a cylinder |
| $\mu$ | Poisson's ratio for plastic deformation of a face material |
| $\mu^*$ | generalized form of Poisson's ratio |
| $\nu$ | Poisson's ratio for elastic deformation of a face material |
| $\rho$ | ratio of shear moduli of core $(\sqrt{G_2/G_1})$ ; radius of curvature |
| $\rho_1, \rho_2$ | principal radii of curvature of a surface |

| | |
|---|---|
| $\rho_s, \rho_n$ | radii of curvature in the s and n directions of a reference surface |
| $\sigma_1, \sigma_2, \tau_{12}$ | plane stress components in a face layer |
| $\sigma_3, \tau_{13}, \tau_{23}$ | antiplane stress components in a core layer |
| $\tau$ | twist of a surface |
| $\xi_1, \xi_2$ | orthogonal curvilinear coordinates on a reference surface (along curvature directions) |
| $\xi_s, \xi_n$ | orthogonal coordinates on reference surface ($\xi_s$ is taken along the boundary of the region occupied by the shell) |
| $\varphi$ | stiffness parameter $\left(= \frac{2}{\pi^4} \frac{Fc^3}{D}\right)$ ; circumferential coordinate for a cylinder |
| $\varphi^*$ | reduced stiffness parameter $\left(= \frac{F^*}{F} \frac{\bar{E}}{\bar{E}^*} \varphi\right)$ |
| $\omega_n, \omega_s$ | rotations of the normal to the middle surface of a face layer (referred to n and s coordinate direction at the edge of a shell) |
| $\omega_1, \omega_2$ | rotations of the normal to the middle surface of a face layer (referred to 1 and 2 coordinate directions) |
| $(\ )_{,x}, _{,xx}$, etc. | $\partial/\partial x$ , $\partial^2/\partial x^2$ , etc. |
| $_{,\varphi}, _{,\varphi\varphi}$, etc. | $\frac{1}{R_i} \frac{\partial}{\partial \varphi}$ , $\frac{1}{R_i^2} \frac{\partial^2}{\partial \varphi^2}$ , etc. (i=1,2) |
| $_{,\bar{\varphi}}$ | $\frac{1}{R} \frac{\partial}{\partial \varphi}$ |
| $_{,\xi}$ | $\frac{\partial}{\partial \xi}$ |
| $\nabla^2$ | Laplace operator for two surface coordinates |
| $\nabla^4, \nabla^6, \ldots$ | $\nabla^2\nabla^2, \nabla^2\nabla^2\nabla^2, \ldots$ |

SUBSCRIPTS

| | |
|---|---|
| a | axisymmetric |
| c | core |
| cl | classical |
| cr | critical |
| f | face |
| i | integer (1 or 2) |
| w | wrinkling |

# INTRODUCTION

Although substantial interest in sandwich construction for aeronautical applications was evidenced during and just subsequent to World War II, it has been only in the last few years that the inherent potential of composite construction, in general, has been truly appreciated. The continual striving for lightweight structures with higher strength and stiffness has caused significant interest in composites for aerospace structures applications and, as a result, has led to renewed interest in the sandwich type of composite. More and more aerospace designs are reflecting the usage of sandwich construction in primary structural members as fabrication and prediction capabilities improve.

As is usually the case in aerospace developments, engineering achievements outstrip theoretical prediction of performance. Thus, to avoid costly experimental design and conservative analysis techniques and to realize the potential of composite structures in competition with conventional metal structures, theory and practice must be made concurrent. Unfortunately, the price to be paid for achieving more optimum structures is complexity. Since the inherent complexity manifests itself in the fabrication, testing, and analysis of sandwich construction, each of the areas must receive comparable attention and the problems must be attacked directly rather than accommodated by the introduction of conservative practices. The status of the design criteria for stiffened metal shells attests to the penalties incurred by the absence of a satisfactory theoretical solution of the problem, even though the fabrication and testing of such shells is very well in hand.

A perusal of Reference 1, a very extensive study of the history and state of the art in the establishment of sandwich construction design criteria through 1965, indicates that the various stress, deflection, and stability analyses of sandwich beams, plates, and shells are of a discrete, rather than a unified, nature and are somewhat limited in scope. For example, no single analysis is referred to which is sufficiently general to predict simultaneously both general instability and face wrinkling of sandwich plates or shells.

Since the compilation of the material presented in Reference 1, unified studies have been completed. References 2 and 3, which are independent and different approaches to the same basic problem, deal with sandwich plates in which both transverse shear and normal strain effects are included. In the former, a variational approach is utilized and both governing equations and consistent boundary conditions for prescribing either forces or displacements are presented. Applications are made to the stability in axial compression of two types of edge-supported sandwich plates in References 2 and 4. In Reference 3, governing equations are presented for sandwich plates with dissimilar faces based on equilibrium and continuity conditions developed in Reference 5; force boundary conditions, established in Reference 5 through virtual work considerations, are listed. An application to buckling in axial compression and bending of sandwich columns with dissimilar faces results in obviously incorrect design curves. In addition, for application to plates (all edges supported), the buckling equations are not consistent with the number of boundary conditions prescribed. Wempner et al., in References 5 and 6, treat the problem of sandwich shells, the first being a general theory and the second a specialization of the general theory to the specific problem of a thin cylindrical shell undergoing moderately large deflections. However, when the buckling of shells is treated, assumptions are invoked in References 5 and 6 which render the buckling equations unusable for studying all the wrinkling modes of instability. Also, in Reference 5, the core-constitutive equations are unduly complex relative to the core model employed. In connection with the work of Wempner et al., it must be remarked that, despite earlier presentation, its very late availability in the general literature (1965) made the theoretical development of Reference 5 (the foundation of Reference 3) unknown to the authors during the development of the present theory.

The present study has been undertaken (1) as a logical extension to curved plates and shells of the unified theory of Reference 2, (2) to present a set of governing equations, together with consistent force or displacement boundary conditions, for the buckling and bending of curved sandwich plates and shells, and (3) to correlate the theoretical predictions for the buckling of honeycomb-core sandwich shells in axial compression with test data available in the literature.

In connection with item (1), the theory of Reference 2 is obtained as a special case when both initial-curvature and moderately large-rotation effects are omitted.

Relative to item (2), the development and form of the governing equations and boundary conditions of the present theory are much less complex than those of References 3 and 5 due to the notation, structural models, and method utilized.

With regard to item (3), there is strong evidence to suggest that the axial compression buckling load of a sandwich shell of practical construction and dimensions, unlike the homogeneous, isotropic shell, is given in good approximation by the linear theory. The discrepancies that occur may be as much a function of the changes in honeycomb-core properties present at the instant of buckling as they are a function of initial imperfections in the geometry of the shell. Thus, a more complete investigation of the problem suggests consideration of the discrete behavior of the honeycomb cells rather than the current procedure of treating the honeycomb core as a continuum.

## THEORY

### GENERAL CONSIDERATIONS

The basic approach taken in this study is to establish equilibrium equations and consistent boundary conditions with the aid of variational procedures. The equilibrium and boundary condition equations for the core and faces (Figure 1) are derived variationally from the condition for a stationary character of the separate total potentials for the face layers and the core layer. The equilibrium equations for the core, applicable in the core volume, can be integrated through the thickness of the core. Then, by enforcing continuity of displacements at the interfaces between the core layer and the face layers, a final set of governing equations and consistent boundary conditions is obtained in only two surface coordinates, comparable to the usual two-dimensional formulation associated with thin shell theory.

### FACE CONSIDERATIONS

The face layers are treated as thin, isotropic shells on the basis of the Kirchhoff-Love assumptions. Rotations about the normal to the middle surface are neglected with respect to rotations about directions tangent to the middle surface.

The middle surface strain displacement relations and the curvature changes are (Reference 7)

$$\epsilon_1 = \frac{1}{\alpha_1} u_{,\xi_1} + \frac{v}{\alpha_1\alpha_2} \alpha_{1,\xi_2} + \frac{w}{\rho_1} + \frac{1}{2}\omega_1^2 \tag{1a}$$

$$\epsilon_2 = \frac{1}{\alpha_1} v_{,\xi_2} + \frac{u}{\alpha_1\alpha_2} \alpha_{2,\xi_1} + \frac{w}{\rho_2} + \frac{1}{2}\omega_2^2 \tag{1b}$$

$$\gamma_{12} = \frac{1}{\alpha_1} v_{,\xi_1} + \frac{1}{\alpha_2} u_{,\xi_2} - \frac{u}{\alpha_1\alpha_2} \alpha_{1,\xi_2} - \frac{v}{\alpha_1\alpha_2} \alpha_{2,\xi_1} + \omega_1\omega_2 \tag{1c}$$

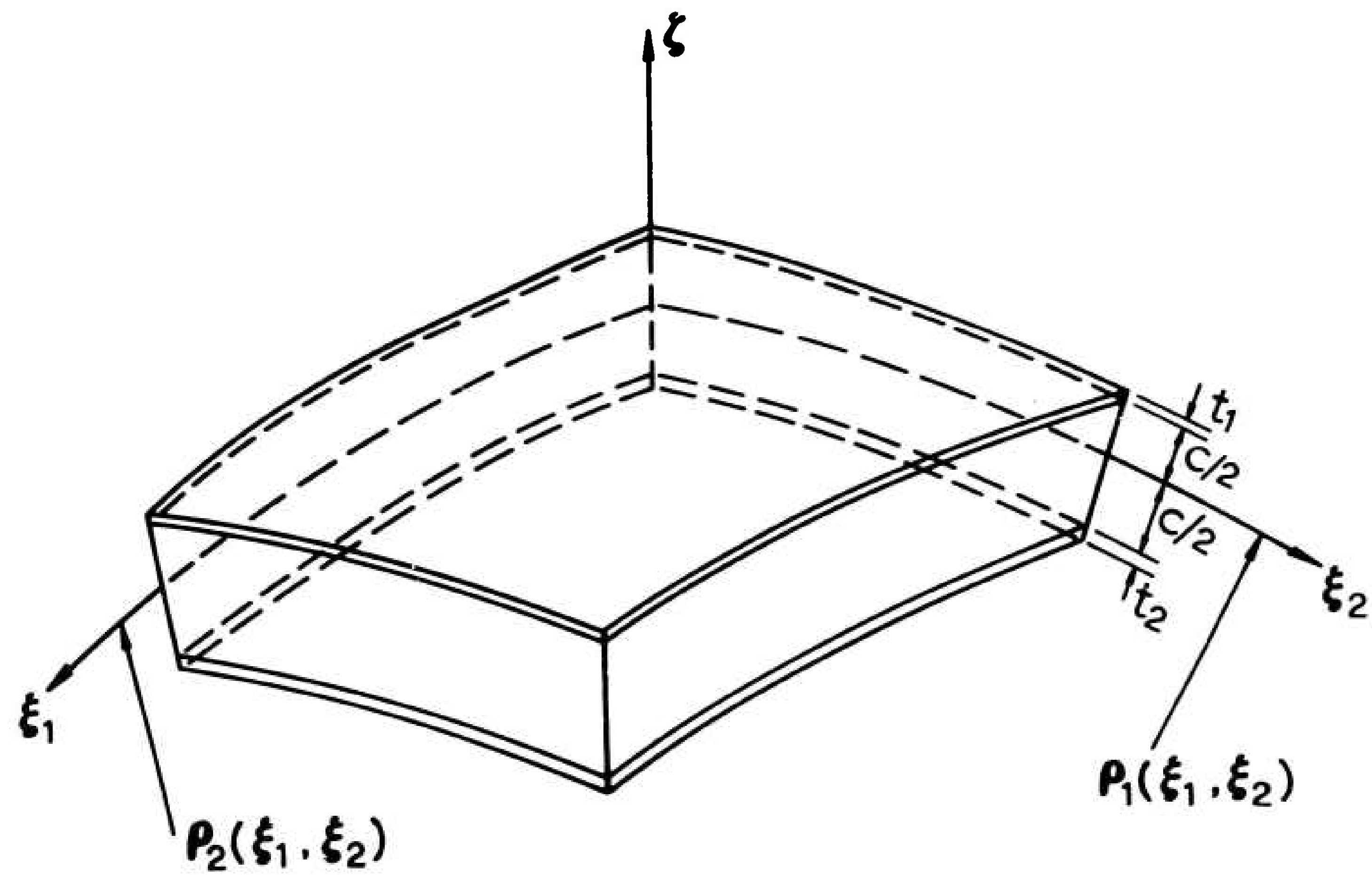


Figure 1. Element of Sandwich Shell.

$$\kappa_1 = \frac{1}{\alpha_1}\,\omega_{1,\xi_1} + \frac{\omega_2}{\alpha_1\alpha_2}\,\alpha_{1,\xi_2} \tag{2a}$$

$$\kappa_2 = \frac{1}{\alpha_2}\,\omega_{2,\xi_2} + \frac{\omega_1}{\alpha_1\alpha_2}\,\alpha_{2,\xi_1} \tag{2b}$$

$$2\kappa_{12} = \frac{1}{\alpha_1}\,\omega_{2,\xi_1} + \frac{1}{\alpha_2}\,\omega_{1,\xi_2} - \frac{\omega_1}{\alpha_1\alpha_2}\,\alpha_{1,\xi_2} - \frac{\omega_2}{\alpha_1\alpha_2}\,\alpha_{2,\xi_1} \tag{2c}$$

where

$$\omega_1 = \frac{1}{\alpha_1}\,w_{,\xi_1} - \frac{u}{\rho_1} \quad \text{and} \quad \omega_2 = \frac{1}{\alpha_2}\,w_{,\xi_2} - \frac{v}{\rho_2}$$

As observed in Appendix I, the strain energy of a linearly elastic isotropic medium for which the Kirchhoff-Love approximations are valid is

$$U_{f_i} = \iint_S \left\{ \frac{C_i}{2}\left(\epsilon_1^2 + \epsilon_2^2 + 2\nu\,\epsilon_1\epsilon_2 + \frac{1-\nu}{2}\,\gamma_{12}^2\right)_i + \frac{D_i}{2}\left[\kappa_1^2 + \kappa_2^2 + 2\nu\,\kappa_1\kappa_2 + 2(1-\nu)\,\kappa_{12}^2\right]_i \right\} (\alpha_1\alpha_2)_i\, d\xi_1 d\xi_2 \tag{3}$$

where

$$C_i = \left[\frac{Et}{1-\nu^2}\right]_i \quad , \quad D_i = \left[\frac{Et^3}{12(1-\nu^2)}\right]_i$$

and $i$ takes on the values 1 and 2 for the upper and lower faces, respectively.

For the edge loads (Figure 2) $N_s$ , $N_n$ , $N_z$ , $M_n$ , and $M_{ns}$ ; the lateral surface load $p$ ; and the interface loads $q_1$ , $q_2$ , and $q_3$ ,

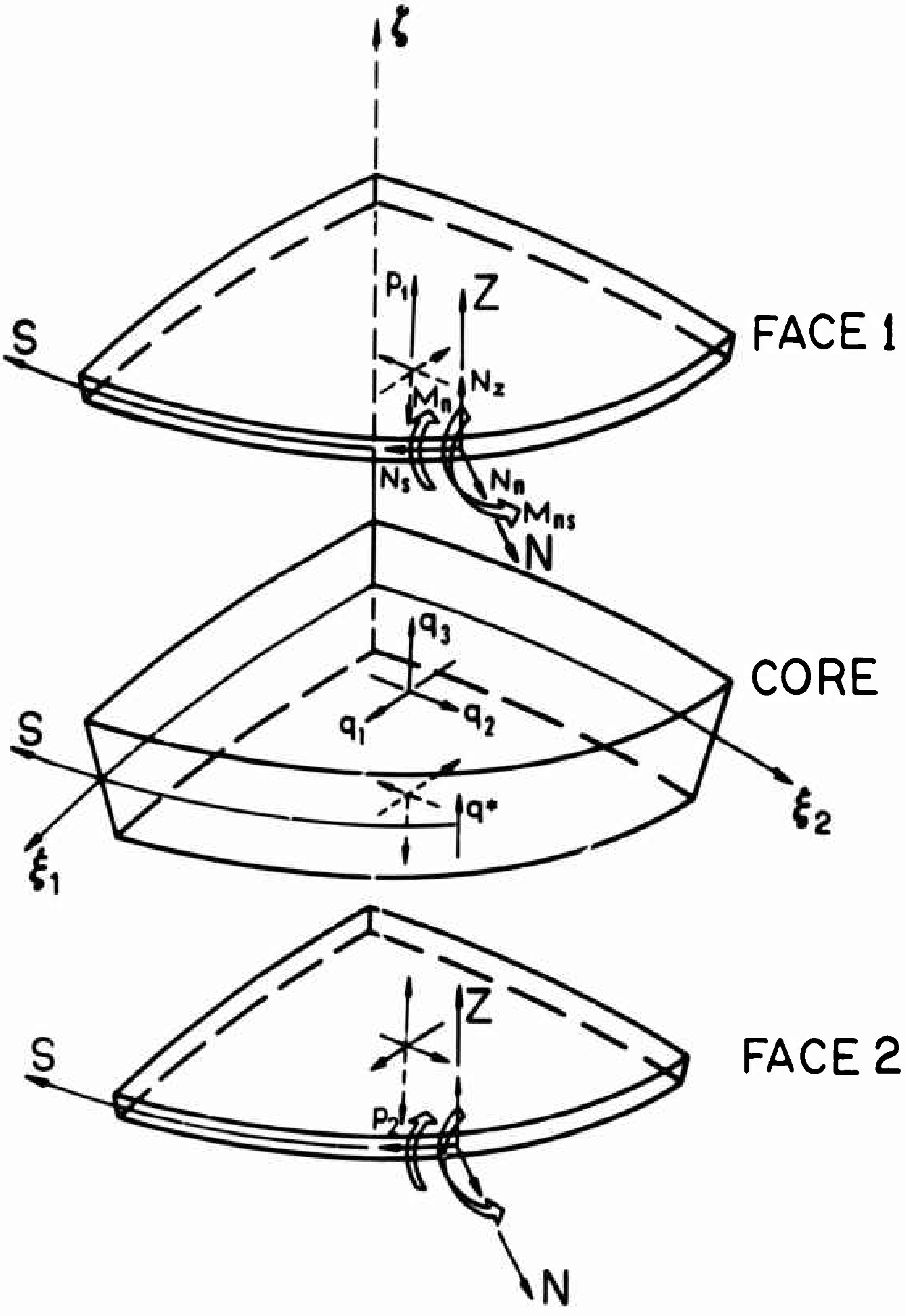


Figure 2. External and Interface Loads on Layers of Sandwich Shell.

the potential of applied loads for one of the faces is

$$V_{f_i} = - \oint_{\partial S} \left\{ \left[ N_n d_n + N_s d_s + N_z w + M_n \omega_n - M_{ns} \omega_s \right] \alpha_s \right\}_i d\xi_s$$

$$\pm \iint_S \left\{ \left[ q_1 \left( u \pm \frac{t}{2} \omega_1 \right) \pm q_2 \left( v \pm \frac{t}{2} \omega_2 \right) \right. \right.$$

$$\left. \left. + \left( q_3 - p \right) w \right] \alpha_1 \alpha_2 \right\}_i d\xi_1 d\xi_2 \tag{4}$$

where the upper sign applies to face 1 ($i = 1$), and the lower sign applies to face 2 ($i = 2$).

Application of the minimum total potential energy principle requires that

$$\delta \left[ U_f + V_f \right]_i \equiv 0 \tag{5}$$

The actual variation is carried out in Appendix I. The resulting Euler equations of equilibrium are

$$\left[ \left\{ \alpha_2 C(\epsilon_1 + \nu\epsilon_2) \right\}_{,\xi_1} - C(\epsilon_2 + \nu\epsilon_1)\, \alpha_{2,\xi_1} + \left\{ \alpha_1 C \frac{1-\nu}{2} \gamma_{12} \right\}_{,\xi_2} \right.$$

$$+ C \frac{1-\nu}{2} \gamma_{12} \alpha_{1,\xi_2} - \frac{1}{\rho_1} \left[ \left\{ \alpha_2 D(\kappa_1 + \nu\kappa_2) \right\}_{,\xi_1} \right.$$

$$- D(\kappa_2 + \nu\kappa_1)\, \alpha_{2,\xi_1} + \left\{ \alpha_1 D(1-\nu)\, \kappa_{12} \right\}_{,\xi_2}$$

$$\left. \left. + D(1-\nu)\, \kappa_{12} \alpha_{1,\xi_2} \right] \mp \alpha_1 \alpha_2 q_1 \right]_i = 0 \tag{6a}$$

$$\Bigg[\left\{\alpha_1 \, C(\epsilon_2 + \nu\epsilon_1)\right\}_{,\xi_2} - C(\epsilon_1 + \nu\epsilon_2)\,\alpha_{1,\xi_2} + \left\{\alpha_2 \, C\,\frac{1-\nu}{2}\,\gamma_{12}\right\}_{,\xi_1}$$

$$+ C\,\frac{1-\nu}{2}\,\gamma_{12}\alpha_{2,\xi_1} - \frac{1}{\rho_2}\Bigg[\left\{\alpha_1 \, D(\kappa_2 + \nu\kappa_1)\right\}_{,\xi_2}$$

$$- D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2} + \left\{\alpha_2 \, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_1}$$

$$+ D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1}\Bigg] \mp \alpha_1\alpha_2 q_2\Bigg]_i = 0 \qquad (6b)$$

$$\Bigg[\left\{\frac{1}{\alpha_1}\Bigg[\left\{\alpha_2 \, D(\kappa_1 + \nu\kappa_2)\right\}_{,\xi_1} - D(\kappa_2 + \nu\kappa_1)\,\alpha_{2,\xi_1} + \left\{\alpha_1 \, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_2}\right.$$

$$\left. + D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2}\Bigg]\right\}_{,\xi_1}$$

$$+ \left\{\frac{1}{\alpha_2}\Bigg[\left\{\alpha_1 \, D(\kappa_2 + \nu\kappa_1)\right\}_{,\xi_2}\right.$$

$$- D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2}$$

$$\left. + \left\{\alpha_2 \, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_1} + D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1}\Bigg]\right\}_{,\xi_2}$$

$$+ \alpha_1\alpha_2 \Bigg[C(\epsilon_1 + \nu\epsilon_2)\left(\frac{1}{\rho_1} - \kappa_1\right)$$

$$+ C(\epsilon_2 + \nu\epsilon_1)\left(\frac{1}{\rho_2} - \kappa_2\right) - 2C\,\frac{1-\nu}{2}\,\gamma_{12}\kappa_{12}$$

$$\pm (q_3 - p)\Bigg] - \frac{t}{2}\left\{(\alpha_2 q_1)_{,\xi_1}\right.$$

$$\left. + (\alpha_1 q_2)_{,\xi_2}\right\}\Bigg]_i = 0 \qquad (6c)$$

The boundary conditions consistent with the equilibrium equations are

$$\Bigg[\Big\{C(\epsilon_1 + \nu\epsilon_2)\, c_{21} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{11} - \frac{D}{\rho_1}\,(1-\nu)\,\kappa_{12}c_{11} - \frac{D}{\rho_1}\,(\kappa_1 + \nu\kappa_2)\, c_{21}\Big\} c_{11}$$

$$+ \Big\{C(\epsilon_2 + \nu\epsilon_1)\, c_{11} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{21} - \frac{D}{\rho_2}\,(1-\nu)\,\kappa_{12}c_{21}$$

$$- \frac{D}{\rho_2}\,(\kappa_2 + \nu\kappa_1)\, c_{11}\Big\} c_{21} + N_s - \frac{M_n}{\tau} + \frac{M_{ns}}{\rho_s}\Bigg]_i = 0$$

$$\text{or} \qquad \delta d_{si} = 0 \tag{7a}$$

$$\Bigg[\Big\{C(\epsilon_1 + \nu\epsilon_2)\, c_{21} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{11} - \frac{D}{\rho_1}\,(1-\nu)\,\kappa_{12}c_{11} - \frac{D}{\rho_1}\,(\kappa_1 + \nu\kappa_2)\, c_{21}\Big\} c_{12}$$

$$+ \Big\{C(\epsilon_2 + \nu\epsilon_1)\, c_{11} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{21} - \frac{D}{\rho_2}\,(1-\nu)\,\kappa_{12}c_{21}$$

$$- \frac{D}{\rho_2}\,(\kappa_2 + \nu\kappa_1)\, c_{11}\Big\} c_{22} + N_n - \frac{M_n}{\rho_n} + \frac{M_{ns}}{\tau}\Bigg]_i = 0$$

$$\text{or} \qquad \delta d_{ni} = 0 \tag{7b}$$

$$\Bigg[\Big\{C(\epsilon_1 + \nu\epsilon_2)\,\omega_1 + C\,\frac{1-\nu}{2}\,\gamma_{12}\omega_2\Big\} c_{21} + \Big\{C(\epsilon_2 + \nu\epsilon_1)\,\omega_2 + C\,\frac{1-\nu}{2}\,\gamma_{12}\omega_1\Big\} c_{11}$$

$$- \frac{1}{\alpha_s}\Big[d_{11}\Big\{D(\kappa_1 + \nu\kappa_2)\, c_{21} + D(1-\nu)\,\kappa_{12}c_{11}\Big\}$$

$$+ d_{12}\Big\{D(\kappa_2 + \nu\kappa_1)\, c_{11} + D(1-\nu)\,\kappa_{12}c_{21}\Big\}\Big]_{,\xi_s} \qquad \text{(Cont'd)}$$

$$- \frac{1}{\alpha_1\alpha_2}\left[\left\{\alpha_2\, D(\kappa_1 + \nu\kappa_2)\right\}_{,\xi_1} - D(\kappa_2 + \nu\kappa_1)\,\alpha_{2,\xi_1}\right.$$

$$\left. + \left\{\alpha_1\, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_2} + D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2}\right] c_{21}$$

$$- \frac{1}{\alpha_1\alpha_2}\left[\left\{\alpha_1\, D(\kappa_2 + \nu\kappa_1)\right\}_{,\xi_2} - D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2}\right.$$

$$\left. + \left\{\alpha_2\, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_1} + D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1}\right] c_{11}$$

$$\left. + \frac{t}{2}\left[q_1 c_{21} + q_2 c_{11}\right] - N_z + \frac{1}{\alpha_s} M_{ns,\xi_s}\right]_i = 0$$

or $\quad \delta w_i = 0$ (7c)

$$\left[\left\{D(\kappa_1 + \nu\kappa_2)\, c_{21} + D(1-\nu)\,\kappa_{12}c_{11}\right\} d_{21}\right.$$

$$\left. + \left\{D(\kappa_2 + \nu\kappa_1)\, c_{11} + D(1-\nu)\,\kappa_{12}c_{21}\right\} d_{22} + M_n\right]_i = 0$$

or $\quad \delta w_{i,\xi_n} = 0$ (7d)

where $c_{11}$, $c_{12}$, $d_{11}$, $d_{12}$, etc., are transformation symbols utilized to express the boundary conditions in terms of generalized displacement components corresponding to a curvilinear coordinate system along and normal to the edge of the shell.

## CORE CONSIDERATIONS

The core description utilized herein is representative of that associated with honeycomb-sandwich construction. For such a core, it is reasonable to assume that:

1. The inplane shear and extensional stiffnesses are negligible with respect to transverse stiffness quantities, corresponding to a state of stress in which $\sigma_1 = \sigma_2 = \tau_{12} = 0$ .
2. In view of item 1, it is justifiable to linearize the relevant strain-displacement relations for the core.
3. The transverse stiffnesses per cell are constant through the thickness.

The first assumption is obvious in the case of a free core of cellular structure. When the core is supported by faceplates, the assumption is reasonable for thick cores $(c > a)$ and for laterally loaded sandwich panels with relatively low inplane loads. However, for very thin panels (for example, $c < \frac{a}{2}$ ) loaded in the plane of the structure, the assumption may need reconsideration, especially when face-wrinkling phenomena are being studied.

An element of the sandwich shell under consideration is shown in Figure 1. The middle surface of the undeformed core $(\zeta = 0)$ is utilized as the reference surface of the shell. Orthogonal curvilinear coordinates $\xi_1$ and $\xi_2$ are taken along the curvature directions. The principal radii of curvature are positive as shown in the figure. The coordinate $\zeta$ is measured normal to the $\xi_1\xi_2$ surface.

On the basis of the core model, the relevant constitutive equations for the resulting antiplane state of stress are

$$\sigma_3 = F(\zeta)\,\epsilon_3 \quad ; \quad \tau_{13} = G_1(\zeta)\,\gamma_{13} \quad ; \quad \tau_{23} = G_2(\zeta)\,\gamma_{23} \qquad (8)$$

In Equation (8), the stiffness functions for a curved element are appropriately designated as

$$F(\zeta) = F\,\frac{\bar{\alpha}_1\bar{\alpha}_2}{\alpha_1(\zeta)\,\alpha_2(\zeta)} \qquad \text{(Cont'd)}$$

$$G_1(\zeta) = G_1 \frac{\bar{\alpha}_2}{\alpha_2(\zeta)}$$

$$G_2(\zeta) = G_2 \frac{\bar{\alpha}_1}{\alpha_1(\zeta)} \tag{9}$$

where $\bar{\alpha}_1$ , $\bar{\alpha}_2$ , F , $G_1$ , and $C_2$ are the values of these quantities when $\zeta = 0$ .

The linearized strain displacement relations are

$$\gamma_{13} = u_{,\zeta} - \frac{u}{\alpha_1} \alpha_{1,\zeta} + \frac{1}{\alpha_1} w_{,\xi_1} \tag{10a}$$

$$\gamma_{23} = v_{,\zeta} - \frac{v}{\alpha_2} \alpha_{2,\zeta} + \frac{1}{\alpha_2} w_{,\xi_2} \tag{10b}$$

$$\epsilon_3 = w_{,\zeta}$$

With the use of Equations (8) and (10), it is shown in Appendix II that the strain energy of the core is

$$U_c = \iiint_R \left[\frac{1}{2} F(\zeta)\, \epsilon_3^2 + \frac{1}{2} G_1(\zeta)\, \gamma_{13}^2 + \frac{1}{2} G_2(\zeta)\, \gamma_{23}^2\right] \alpha_1\alpha_2 d\xi_1 d\xi_2 d\zeta \tag{11}$$

For the edge load $q^*$ and the interface load system with components $q_1$ , $q_2$ , and $q_3$ (Figure 2), the potential of applied loads is

$$V_c = - \iint_S \left[(q_1 u + q_2 v + q_3 w)\, \alpha_1\alpha_2\right]_{\zeta=-(c/2)}^{c/2} d\xi_1 d\xi_2$$

$$- \oint_{\partial S} \int_{-(c/2)}^{c/2} q^* w \alpha_s d\xi_s d\zeta \tag{12}$$

When the minimum total potential energy principle is applied to the core in the form

$$\delta \left[ U_c + V_c \right] \equiv 0 \tag{13}$$

the equilibrium equations and boundary conditions are obtained as shown in Appendix II. The equilibrium equations are

$$G_1 \frac{\bar{\alpha}_2}{\alpha_1} (\alpha_1^2 \gamma_{13})_{,\zeta} = 0 \tag{14a}$$

$$G_2 \frac{\bar{\alpha}_1}{\alpha_2} (\alpha_2^2 \gamma_{23})_{,\zeta} = 0 \tag{14b}$$

$$w_{,\zeta\zeta} + \frac{1}{F\bar{\alpha}_1\bar{\alpha}_2} (\bar{\alpha}_2 G_1 \gamma_{13})_{,\xi_1} + (\bar{\alpha}_1 G_2 \gamma_{23})_{,\xi_2} = 0 \tag{14c}$$

The boundary conditions consistent with these equilibrium equations are

$$G_1 \frac{\bar{\alpha}_2}{\alpha_2} \gamma_{13} = q_1 \tag{15a}$$

$$G_2 \frac{\bar{\alpha}_1}{\alpha_1} \gamma_{23} = q_2 \tag{15b}$$

$$F \frac{\bar{\alpha}_1\bar{\alpha}_2}{\alpha_1\alpha_2} w_{,\zeta} = q_3 \tag{15c}$$

$$G_1 \frac{\bar{\alpha}_2}{\alpha_2} \gamma_{13}\alpha_2 d\xi_2 + G_2 \frac{\bar{\alpha}_1}{\alpha_1} \gamma_{23}\alpha_1 d\xi_1 = - q^* \alpha_s d\xi_s \quad \text{or} \quad \delta w = 0 \tag{15d}$$

INTERFACE CONTINUITY CONSIDERATIONS

At the present point, the governing equations and boundary conditions have been derived for the core and faces considered as separate media.

In order to ensure continuity of deformation in the composite sandwich shell, the components of the core and face displacements must satisfy the conditions

$$u(\xi_1,\xi_2,\tfrac{c}{2}) = (u + \tfrac{t}{2}\,\omega_1)_1$$
$$u(\xi_1,\xi_2,-\tfrac{c}{2}) = (u - \tfrac{t}{2}\,\omega_1)_2 \qquad (16a)$$

$$v(\xi_1,\xi_2,\tfrac{c}{2}) = (v + \tfrac{t}{2}\,\omega_2)_1$$
$$v(\xi_1,\xi_2,-\tfrac{c}{2}) = (v - \tfrac{t}{2}\,\omega_2)_2 \qquad (16b)$$

$$w(\xi_1,\xi_2,\tfrac{c}{2}) = w_1$$
$$w(\xi_1,\xi_2,-\tfrac{c}{2}) = w_2 \qquad (16c)$$

As demonstrated in Appendix II, the satisfaction of conditions (16a,b, and c) subsequent to integration of the core strain-displacement equations gives rise to the two interface compatibility equations

$$u_1\left(1 - \frac{c}{2\rho_1}\right) - u_2\left(1 + \frac{c}{2\rho_1}\right) + \frac{1}{\bar{\alpha}_1}\, w_{1,\xi_1}\left[\frac{c + t_1}{2} - \frac{c^2 + 3t_1c}{6\rho_1}\right]$$
$$+ \frac{1}{\bar{\alpha}_1}\, w_{2,\xi_1}\left[\frac{c + t_2}{2} + \frac{c^2 + 3t_2c}{6\rho_1}\right] - \gamma_1 c$$
$$+ \frac{c^3}{12\,F\bar{\alpha}_1}\left[\frac{G_1}{\bar{\alpha}_1\bar{\alpha}_2}\,(\bar{\alpha}_2\gamma_1)_{,\xi_1} + \frac{G_2}{\bar{\alpha}_1\bar{\alpha}_2}\,(\bar{\alpha}_1\gamma_2)_{,\xi_2}\right]_{,\xi_1} = 0 \qquad (17)$$

and

$$
v_1\left(1-\frac{c}{2\rho_2}\right)-v_2\left(1+\frac{c}{2\rho_2}\right)+\frac{1}{\bar{\alpha}_2}\,w_{1,\xi_2}\left[\frac{c+t_1}{2}-\frac{c^2+3t_1c}{6\rho_2}\right]
$$

$$
+\frac{1}{\bar{\alpha}_2}\,w_{2,\xi_2}\left[\frac{c+t_2}{2}+\frac{c^2+3t_2c}{6\rho_2}\right]-\gamma_2 c
$$

$$
+\frac{c^3}{12\,F\bar{\alpha}_2}\left[\frac{G_1}{\bar{\alpha}_1\bar{\alpha}_2}\left(\bar{\alpha}_2\gamma_1\right)_{,\xi_1}+\frac{G_2}{\bar{\alpha}_1\bar{\alpha}_2}\left(\bar{\alpha}_1\gamma_2\right)\right]_{,\xi_2}=0 \qquad (18)
$$

As is also shown in Appendix II, additional results stemming from the integration of the core equations permit the interface loads $q_1$, $q_2$, and $q_3$ and the boundary conditions for the edge of the core to be expressed in terms of the face displacement components $u_i$, $v_i$, and $w_i$ and the core shear angles $\gamma_1$ and $\gamma_2$.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

With terms usually eliminated from shallow-shell theory denoted by an asterisk, the final set of governing equations describing the behavior of the sandwich shell is

$$
\Bigg[\left\{\alpha_2\,C(\epsilon_1+\nu\epsilon_2)\right\}_{,\xi_1}-C(\epsilon_2+\nu\epsilon_1)\,\alpha_{2,\xi_1}+\left\{\alpha_1\,C\,\frac{1-\nu}{2}\,\gamma_{12}\right\}_{,\xi_2}
$$

$$
+C\,\frac{1-\nu}{2}\,\gamma_{12}\alpha_{1,\xi_2}-\frac{1}{\rho_1}\Bigg[\left\{\alpha_2\,D(\kappa_1+\nu\kappa_2)\right\}_{,\xi_1}
$$

$$
-D(\kappa_2+\nu\kappa_1)\,\alpha_{2,\xi_1}+\left\{\alpha_1\,D(1-\nu)\,\kappa_{12}\right\}_{,\xi_2}
$$

$$
+D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2}\Bigg]^{*}
$$

$$
\mp\,\alpha_1\alpha_2G_1\gamma_1\left(1\mp c\,\frac{\rho_1+2\rho_2}{2\rho_1\rho_2}\right)\Bigg]_i=0 \qquad (19a)
$$

$$\left[\left\{\alpha_1\, C(\epsilon_2 + \nu\epsilon_1)\right\}_{,\xi_2} - C(\epsilon_1 + \nu\epsilon_2)\,\alpha_{1,\xi_2} + \left\{\alpha_2\, C\,\frac{1-\nu}{2}\,\gamma_{12}\right\}_{,\xi_1}\right.$$

$$+ C\,\frac{1-\nu}{2}\,\gamma_{12}\alpha_{2,\xi_1} - \frac{1}{\rho_2}\left[\left\{\alpha_1\, D(\kappa_2 + \nu\kappa_1)\right\}_{,\xi_2}\right.$$

$$- D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2} + \left\{\alpha_2\, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_1}$$

$$\left. + D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1}\right]^{*}$$

$$\left.\mp\, \alpha_1\alpha_2 G_2\gamma_2\left(1 \mp c\,\frac{\rho_2 + 2\rho_1}{2\rho_1\rho_2}\right)\right]_i = 0 \qquad (19b)$$

$$\left[\left\{\frac{1}{\alpha_1}\left[\left\{\alpha_2\, D(\kappa_1 + \nu\kappa_2)\right\}_{,\xi_1} - D(\kappa_2 + \nu\kappa_1)\,\alpha_{2,\xi_1} + \left\{\alpha_1\, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_2}\right.\right.\right.$$

$$\left.\left. + D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2}\right]\right\}_{,\xi_1}$$

$$+ \left\{\frac{1}{\alpha_2}\left[\left\{\alpha_1\, D(\kappa_2 + \nu\kappa_1)\right\}_{,\xi_2}\right.\right.$$

$$- D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2} + \left\{\alpha_2\, D(1-\nu)\,\kappa_{12}\right\}_{,\xi_1}$$

$$\left.\left. + D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1}\right]\right\}_{,\xi_2}$$

$$+ \alpha_1\alpha_2\left[C(\epsilon_1 + \nu\epsilon_2)\left(\frac{1}{\rho_1} - \kappa_1\right)\right. \qquad \text{(Cont'd)}$$

$$+ C(\epsilon_2 + \nu\epsilon_1)\left(\frac{1}{\rho_2} - \kappa_2\right) - 2C\,\frac{1-\nu}{2}\,\gamma_{12}\kappa_{12}$$

$$\mp p \pm F\left(1 \mp c\,\frac{\rho_1 + \rho_2}{2\rho_1\rho_2}\right)$$

$$\cdot \left\{ \frac{w_1 - w_2}{c} \mp \frac{c}{2}\left(1 \mp \frac{c}{3\rho_1}\right) \frac{G_1}{F\bar{\alpha}_1\bar{\alpha}_2}\,(\bar{\alpha}_2\gamma_1)_{,\xi_1} \right.$$

$$\left. \mp \frac{c}{2}\left(1 \mp \frac{c}{3\rho_2}\right) \frac{G_2}{F\bar{\alpha}_1\bar{\alpha}_2}\,(\bar{\alpha}_1\gamma_2)_{,\xi_2} \right\} \Bigg|$$

$$- \frac{t}{2}\left[\left\{\alpha_2 G_1\gamma_1\left(1 \mp c\,\frac{\rho_1 + 2\rho_2}{2\rho_1\rho_2}\right)\right\}_{,\xi_1}\right.$$

$$\left.\left. + \left\{\alpha_1 G_2\gamma_2\left(1 \mp c\,\frac{2\rho_1 + \rho_2}{2\rho_1\rho_2}\right)\right\}_{,\xi_2}\right]\right]_i = 0 \qquad (19c)$$

The boundary conditions in final form, which permit either generalized forces or displacements to be prescribed, are

$$\Bigg[\left\{C(\epsilon_1 + \nu\epsilon_2)\, c_{21} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{11} - \frac{D}{\rho_1}\,(1-\nu)\,\kappa_{12}c^*_{11} - \frac{D}{\rho_1}\,(\kappa_1 + \nu\kappa_2)\, c^*_{21}\right\} c_{11}$$

$$+ \left\{C(\epsilon_2 + \nu\epsilon_1)\, c_{11} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{21} - \frac{D}{\rho_2}\,(1-\nu)\,\kappa_{12}c^*_{21}\right.$$

$$\left. - \frac{D}{\rho_2}\,(\kappa_2 + \nu\kappa_1)\, c^*_{11}\right\} c_{21} + N_s - \frac{M^*_n}{\tau} + \frac{M^*_{ns}}{\rho_s}\Bigg]_i = 0$$

$$\text{or} \qquad \delta d_{si} = 0 \qquad (20a)$$

$$\left[ \left\{ C(\epsilon_1 + \nu\epsilon_2)\, c_{21} + C\,\frac{1-\nu}{2}\,\gamma_{12} c_{11} - \frac{D}{\rho_1}\,(1-\nu)\,\kappa_{12} c^{*}_{11} - \frac{D}{\rho_1}\,(\kappa_1 + \nu\kappa_2)\, c^{*}_{21} \right\} c_{12} \right.$$

$$+ \left\{ C(\epsilon_2 + \nu\epsilon_1)\, c_{11} + C\,\frac{1-\nu}{2}\,\gamma_{12} c_{21} - \frac{D}{\rho_2}\,(1-\nu)\,\kappa_{12} c^{*}_{21} \right.$$

$$\left. \left. - \frac{D}{\rho_2}\,(\kappa_2 + \nu\kappa_1)\, c^{*}_{11} \right\} c_{22} + N_n - \frac{M^{*}_{n}}{\rho_n} + \frac{M^{*}_{ns}}{\tau} \right]_i = 0$$

or $\delta d_{ni} = 0$ (20b)

$$\left[ \left\{ C(\epsilon_1 + \nu\epsilon_2)\, \omega_1 + C\,\frac{1-\nu}{2}\,\gamma_{12}\omega_2 \right\} c_{21} + \left\{ C(\epsilon_2 + \nu\epsilon_1)\, \omega_2 + \frac{1-\nu}{2}\,\gamma_{12}\omega_1 \right\} c_{11} \right.$$

$$- \frac{1}{\alpha_s} \left[ d_{11} \left\{ D(\kappa_1 + \nu\kappa_2)\, c_{21} + D(1-\nu)\,\kappa_{12} c_{11} \right\} \right.$$

$$\left. + d_{12} \left\{ D(\kappa_2 + \nu\kappa_1)\, c_{11} + D(1-\nu)\,\kappa_{12} c_{21} \right\} \right]_{,\xi_s}$$

$$- \frac{1}{\alpha_1\alpha_2} \left[ \left\{ \alpha_2\, D(\kappa_1 + \nu\kappa_2) \right\}_{,\xi_1} - D(\kappa_2 + \nu\kappa_1)\, \alpha_{2,\xi_1} \right.$$

$$\left. + \left\{ \alpha_1\, D(1-\nu)\,\kappa_{12} \right\}_{,\xi_2} + D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2} \right] c_{21}$$

$$- \frac{1}{\alpha_1\alpha_2} \left[ \left\{ \alpha_1\, D(\kappa_2 + \nu\kappa_1) \right\}_{,\xi_2} - D(\kappa_1 + \nu\kappa_2)\, \alpha_{1,\xi_2} \right.$$

$$\left. + \left\{ \alpha_2\, D(1-\nu)\,\kappa_{12} \right\}_{,\xi_1} + D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1} \right] c_{11}$$

(Cont'd)

$$+ \frac{t}{2} \left[ G_1\gamma_1 \left( 1 \mp c \frac{\rho_1 + 2\rho_2}{2\rho_1\rho_2} \right) c_{21} + G_2\gamma_2 \left( 1 \mp c \frac{\rho_2 + 2\rho_1}{2\rho_1\rho_2} \right) c_{11} \right.$$

$$\left. - N_z + \frac{1}{\alpha_s} M_{ns,\xi_s} \right]_i = 0 \quad \text{or} \quad \delta w_i = 0 \tag{20c}$$

$$\left[ \left\{ D(\kappa_1 + \nu\kappa_2)\, c_{21} + D(1-\nu)\, \kappa_{12} c_{11} \right\} d_{21} \right.$$

$$\left. + \left\{ D(\kappa_2 + \nu\kappa_1)\, c_{11} + D(1-\nu)\, \kappa_{12} c_{21} \right\} d_{22} + M_n \right]_i = 0$$

$$\text{or} \quad \delta w_{i,\xi_n} = 0 \tag{20d}$$

$$c \left[ G_1\gamma_1 c_{21} + G_2\gamma_2 c_{11} + \bar{q}^* \right] = 0 \quad \text{or}$$

$$\delta \left[ w_1 \left( 1 - \frac{c}{3\rho_s} \right) + w_2 \left( 1 + \frac{c}{3\rho_s} \right) + \frac{c^2}{6F} \left\{ \frac{G_1}{\bar{\alpha}_1\bar{\alpha}_2} (\bar{\alpha}_2\gamma_1)_{,\xi_1} \right. \right.$$

$$\left. \left. + \frac{G_2}{\bar{\alpha}_1\bar{\alpha}_2} (\bar{\alpha}_1\gamma_2)_{,\xi_2} \right\} \right] = 0 \tag{20e}$$

The governing equations and boundary conditions given by Equations (17), (18), (19), and (20) are sufficient to determine the bending, buckling, and postbuckling behavior of sandwich shells under prescribed loads and boundary conditions.

Special forms and an application of these equations are treated in sections to follow.

## Shallow Shell Equations

In many instances, the contribution of inplane displacements to the rotations $\omega_1$ and $\omega_2$ of the face layer is small. Especially in the case of shallow-sandwich shells and shallow regions of nonshallow shells, the expressions for the rotations can be simplified to give

$$\begin{aligned} \omega_1 &= \frac{1}{\alpha_1} w_{,\xi_1} \\ \omega_2 &= \frac{1}{\alpha_2} w_{,\xi_2} \end{aligned} \tag{21}$$

If these rotation expressions are utilized, then terms marked by an asterisk have to be omitted from the governing equations and boundary conditions. In such a case, the equations for the faceplates are essentially those commonly referred to as the von Kármán-Donnell type.

## Thin Shell Equations

In cases where the total shell thickness $(c + t_1 + t_2)$ is negligible with respect to each of the principal radii of curvature $(\rho_1,\rho_2)$, terms of the order $(c + t_1 + t_2)/\rho$ or, in view of the Kirchhoff-Love assumption $(t/\rho) \ll 1$, of the order $c/\rho$ can be neglected with respect to unity. As a further consequence, the differences in the metric components $(\alpha_1,\alpha_2)$ and the radii of curvature $(\rho_1,\rho_2)$ of the two face layers must be neglected. With these simplifications, the governing shallow-shell equations and boundary conditions of the previous section reduce to

$$\alpha_2 \left\{ C(\epsilon_1 + \nu\epsilon_2) \right\}_{i\,,\xi_1} - \left\{ C(\epsilon_2 + \nu\epsilon_1) \right\}_i \alpha_{2,\xi_1} + \alpha_1 \left\{ C \frac{1-\nu}{2} \gamma_{12} \right\}_{i\,,\xi_2}$$
$$+ \left\{ C \frac{1-\nu}{2} \gamma_{12} \right\}_i \alpha_{1,\xi_2} \mp \alpha_1\alpha_2 G_1 \gamma_1 = 0 \tag{22a}$$

$$\alpha_1 \left\{ C(\epsilon_2 + \nu\epsilon_1) \right\}_{i\ ,\xi_2} - \left\{ C(\epsilon_1 + \nu\epsilon_2) \right\}_i \alpha_{1,\xi_2} + \alpha_2 \left\{ C \frac{1-\nu}{2} \gamma_{12} \right\}_{i\ ,\xi_1}$$

$$+ \left\{ C \frac{1-\nu}{2} \gamma_{12} \right\}_i \alpha_{2,\xi_1} \mp \alpha_1\alpha_2 G_2 \gamma_2 = 0 \qquad (22b)$$

$$\left\{ \frac{1}{\alpha_1} \left[ \left\{ \alpha_2\ D(\kappa_1 + \nu\kappa_2) \right\}_{i\ ,\xi_1} - \left\{ D(\kappa_2 + \nu\kappa_1) \right\}_i \alpha_{2,\xi_1} + \left\{ \alpha_1\ D(1-\nu)\,\kappa_{12} \right\}_{i\ ,\xi_2} \right.\right.$$

$$\left.\left. + \left\{ D(1-\nu)\ \kappa_{12} \right\}_i \alpha_{1,\xi_2} \right] \right\}_{,\xi_1}$$

$$+ \left\{ \frac{1}{\alpha_2} \left[ \left\{ \alpha_1\ D(\kappa_2 + \nu\kappa_1) \right\}_{i\ ,\xi_2} \right.\right.$$

$$- \left\{ D(\kappa_1 + \nu\kappa_2) \right\}_i \alpha_{1,\xi_2}$$

$$+ \left\{ \alpha_2\ D(1-\nu)\ \kappa_{12} \right\}_{i\ ,\xi_1}$$

$$\left.\left. + \left\{ D(1-\nu)\ \kappa_{12} \right\}_i \alpha_{2,\xi_1} \right] \right\}_{,\xi_2}$$

$$+ \alpha_1\alpha_2 \left\{ C(\epsilon_1 + \nu\epsilon_2) \left( \frac{1}{\rho_1} - \kappa_1 \right) \right.$$

$$\left. + C(\epsilon_2 + \nu\epsilon_1) \left( \frac{1}{\rho_2} - \kappa_2 \right) - 2C \frac{1-\nu}{2} \gamma_{12}\kappa_{12} \mp p \right\}_i$$

$$\pm \alpha_1\alpha_2 \frac{F}{c} (w_1 - w_2)$$

$$- \frac{t_i + c}{2} \left\{ G_1(\alpha_2\gamma_1)_{,\xi_1} \right.$$

$$\left. + G_2(\alpha_1\gamma_2)_{,\xi_2} \right\} = 0 \qquad (22c)$$

$$\left\{C(\epsilon_1 + \nu\epsilon_2)\right\}_i c_{21}c_{11} + \left\{C\,\frac{1-\nu}{2}\,\gamma_{12}\right\}_i \left(c_{11}^2 + c_{21}^2\right)$$

$$+ \left\{C(\epsilon_2 + \nu\epsilon_1)\right\}_i c_{11}c_{21} + N_{si} = 0 \quad \text{or} \quad \delta d_{si} = 0 \qquad \text{(23a)}$$

$$\left\{C(\epsilon_1 + \nu\epsilon_2)\right\}_i c_{21}c_{12} + \left\{C\,\frac{1-\nu}{2}\,\gamma_{12}\right\}_i (c_{11}c_{12} + c_{21}c_{22})$$

$$+ \left\{C(\epsilon_2 + \nu\epsilon_1)\right\}_i c_{11}c_{22} + N_{ni} = 0 \quad \text{or} \quad \delta d_{ni} = 0 \qquad \text{(23b)}$$

$$\left\{C(\epsilon_1 + \nu\epsilon_2)\,\omega_1 + C\,\frac{1-\nu}{2}\,\gamma_{12}\omega_2\right\}_i c_{21}$$

$$+ \left\{C(\epsilon_2 + \nu\epsilon_1)\,\omega_2 + C\,\frac{1-\nu}{2}\,\gamma_{12}\omega_1\right\}_i c_{11}$$

$$- \frac{1}{\alpha_s}\left\{d_{11}\left[\left\{D(\kappa_1 + \nu\kappa_2)\right\}_i c_{21} + \left\{D(1-\nu)\,\kappa_{12}\right\}_i c_{11}\right]\right.$$

$$\left. + d_{12}\left[\left\{D(\kappa_2 + \nu\kappa_1)\right\}_i c_{11} + \left\{D(1-\nu)\,\kappa_{12}\right\}_i c_{21}\right]\right\}_{,\xi_s}$$

$$- \frac{c_{21}}{\alpha_1\alpha_2}\left[\left\{\alpha_2\, D(\kappa_1 + \nu\kappa_2)\right\}_{i\ ,\xi_1} - \left\{D(\kappa_2 + \nu\kappa_1)\right\}_i \alpha_{2,\xi_1}\right.$$

$$\left. + \left\{\alpha_1\, D(1-\nu)\,\kappa_{12}\right\}_{i\ ,\xi_2} + \left\{D(1-\nu)\,\kappa_{12}\right\}_i \alpha_{1,\xi_2}\right]$$

$$- \frac{c_{11}}{\alpha_1\alpha_2}\left[\left\{\alpha_1\, D(\kappa_2 + \nu\kappa_1)\right\}_{i\ ,\xi_2} - \left\{D(\kappa_1 + \nu\kappa_2)\right\}_i \alpha_{1,\xi_2}\right.$$

(Cont'd)

$$+ \left\{\alpha_2 D(1-\nu)\, \kappa_{12}\right\}_{i\;,\xi_1} + \left\{D(1-\nu)\, \kappa_{12}\right\}_i \alpha_{2,\xi_1}\Bigg]$$

$$+ \frac{t_i}{2}\left[G_1\gamma_1 c_{21} + G_2\gamma_2 c_{11}\right] - N_{zi} + \frac{1}{\alpha_s} M_{nsi,\xi_s} = 0$$

or $\delta w_i = 0$ (23c)

$$\left[\left\{D(\kappa_1 + \nu\kappa_2)\right\}_i c_{21} + \left\{D(1-\nu)\, \kappa_{12}\right\}_i c_{11}\right] d_{21}$$

$$+ \left[\left\{D(\kappa_2 + \nu\kappa_1)\right\}_i c_{11} + \left\{D(1-\nu)\, \kappa_{12}\right\}_i c_{21}\right] d_{22}$$

$$+ M_{ni} = 0 \quad \text{or} \quad \delta w_{i,\xi_n} = 0 \tag{23d}$$

$$c\left[G_1\gamma_1 c_{21} + G_2\gamma_2 c_{11} + \overline{q}^*\right] = 0 \quad \text{or}$$

$$\delta\left[w_1 + w_2 + \frac{c^2}{6F\alpha_1\alpha_2}\left\{G_1(\alpha_2\gamma_1)_{,\xi_1} + G_2(\alpha_1\gamma_2)_{,\xi_2}\right\}\right] = 0 \tag{23e}$$

These equations for the sandwich configuration in which $F \rightarrow \infty$ are those of Reference 8. Further, if the bending stiffness of the faces is neglected (membrane faces), then the equations represent the general form of the classical theory for sandwich shells of Reference 9.

## APPLICATION TO COMPRESSIVE BUCKLING OF THIN CIRCULAR CYLINDERS

### GOVERNING EQUATIONS AND BOUNDARY CONSIDERATIONS

The present theory is applied to the problem of determining critical loads for a uniformly axially compressed cylinder of honeycomb sandwich construction. Attention is directed especially to establishing the effect of core stiffness on the distinct instability modes and the interactions of these phenomena. The cylinder is shown in Figure 3.

In Appendix III, the shallow-shell equations are specialized for the case of cylindrical coordinates. Uniform, radially symmetric deformations are assumed to prevail under the axisymmetric load up to a critical value; that is, edge restraints are considered not to influence the deformation prior to buckling. To determine the possible bifurcation points of the load-shortening relationship, infinitesimal deformations measured from the unbuckled, compressed state are used. The specialized governing equations (Equations (119) of Appendix III), in terms of these deformations, are

$$C\left[u_{1,x} + \nu\left(v_{1,\varphi} + \frac{w_1}{R}\right)\right]_{,x} + C\,\frac{1-\nu}{2}\left[u_{1,\varphi} + v_{1,x}\right]_{,\varphi} - G_1\gamma_1 = 0 \qquad (24a)$$

$$C\left[u_{2,x} + \nu\left(v_{2,\varphi} + \frac{w_2}{R}\right)\right]_{,x} + C\,\frac{1-\nu}{2}\left[u_{2,\varphi} + v_{2,x}\right]_{,\varphi} + G_1\gamma_1 = 0 \qquad (24b)$$

$$C\,\frac{1-\nu}{2}\left[u_{1,\varphi} + v_{1,x}\right]_{,x} + C\left[v_{1,\varphi} + \frac{w_1}{R} + \nu u_{1,x}\right]_{,\varphi} - G_2\gamma_2 = 0 \qquad (24c)$$

$$C\,\frac{1-\nu}{2}\left[u_{2,\varphi} + v_{2,x}\right]_{,x} + C\left[v_{2,\varphi} + \frac{w_2}{R} + \nu u_{2,x}\right]_{,\varphi} + G_2\gamma_2 = 0 \qquad (24d)$$

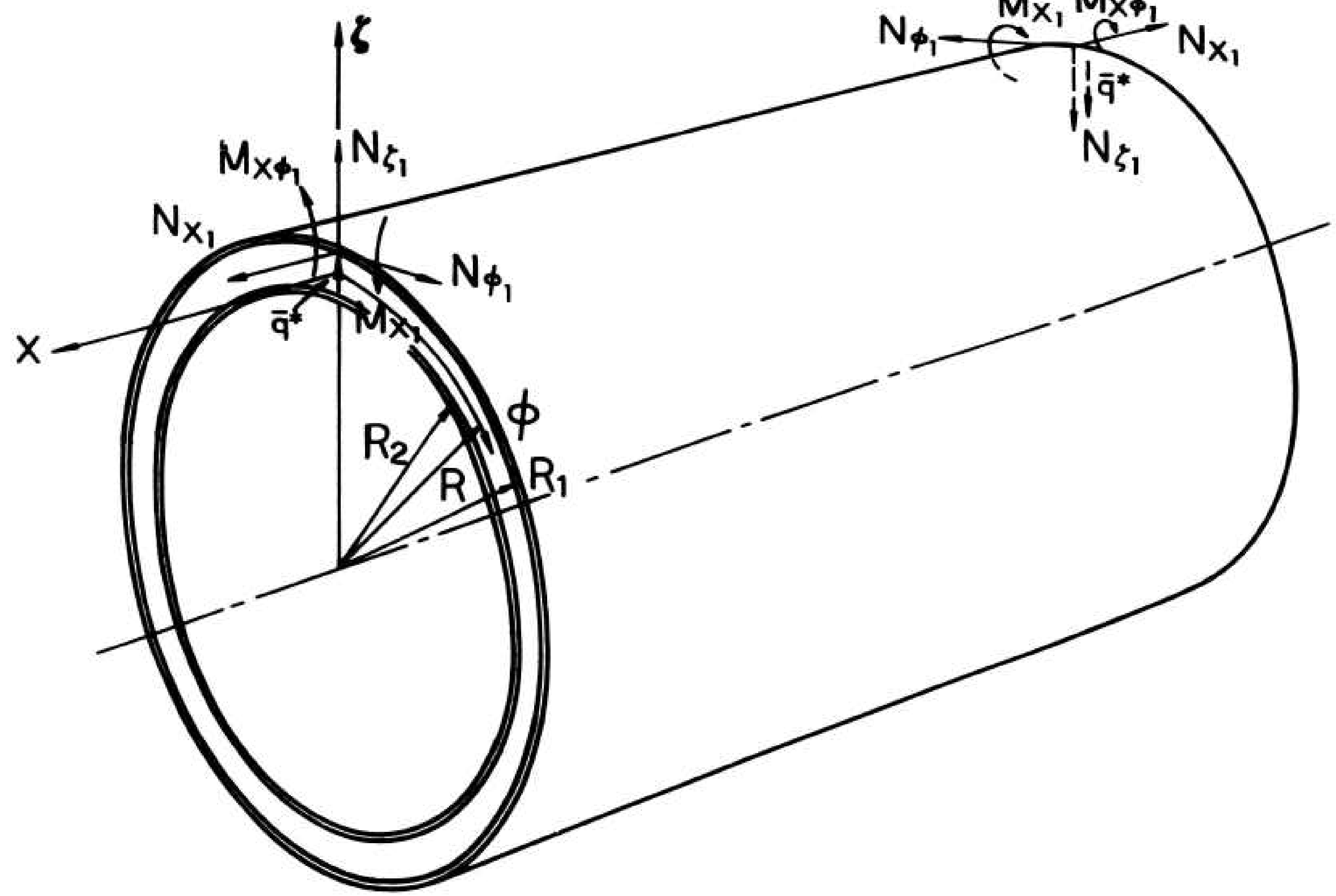


Figure 3. Notation for Circular Sandwich Cylinders.

$$D\nabla^4 w_1 + \frac{C}{R}\left[v_{1,\varphi} + \frac{w_1}{R} + \nu u_{1,x}\right] + N_x w_{1,xx}$$

$$+ F\left[\frac{w_1 - w_2}{c} - \frac{c}{2F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right)\right]$$

$$- \frac{t}{2}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right) = 0 \qquad (24e)$$

$$D\nabla^4 w_2 + \frac{C}{R}\left[v_{2,\varphi} + \frac{w_2}{R} + \nu u_{2,x}\right] + N_x w_{2,xx}$$

$$- F\left[\frac{w_1 - w_2}{c} + \frac{c}{2F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right)\right]$$

$$- \frac{t}{2}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right) = 0 \qquad (24f)$$

$$u_1 - u_2 - c\gamma_1 + \frac{c^3}{12F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right)_{,x} + \frac{c+t}{2}\left(w_1 + w_2\right)_{,x} = 0 \qquad (24g)$$

$$v_1 - v_2 - c\gamma_2 + \frac{c^3}{12F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right)_{,\varphi} + \frac{c+t}{2}\left(w_1 + w_2\right)_{,\varphi} = 0 \qquad (24h)$$

The appropriate boundary conditions for the specialized governing equations (Equation (119) of Appendix III) are

$$C\left[u_{1,x} + \nu\left(v_{1,\varphi} + \frac{w_1}{R}\right)\right] = 0 \quad \text{or} \quad u_1 = 0 \qquad (25a)$$

$$C\left[u_{2,x} + \nu\left(v_{2,\varphi} + \frac{w_2}{R}\right)\right] = 0 \quad \text{or} \quad u_2 = 0 \qquad (25b)$$

$$C\,\frac{1-\nu}{2}\left[u_{1,\varphi} + v_{1,x}\right] = 0 \quad \text{or} \quad v_1 = 0 \tag{25c}$$

$$C\,\frac{1-\nu}{2}\left[u_{2,\varphi} + v_{2,x}\right] = 0 \quad \text{or} \quad v_2 = 0 \tag{25d}$$

$$N_x w_{1,x} - D\left[w_{1,xxx} + (2-\nu)\, w_{1,x\varphi\varphi}\right] + \frac{t}{2}\, G_1 \gamma_1 = 0 \quad \text{or} \quad w_1 = 0 \tag{25e}$$

$$N_x w_{2,x} - D\left[w_{2,xxx} + (2-\nu)\, w_{2,x\varphi\varphi}\right] + \frac{t}{2}\, G_1 \gamma_1 = 0 \quad \text{or} \quad w_2 = 0 \tag{25f}$$

$$D\left[w_{1,xx} + \nu w_{1,\varphi\varphi}\right] = 0 \quad \text{or} \quad w_{1,x} = 0 \tag{25g}$$

$$D\left[w_{2,xx} + \nu w_{2,\varphi\varphi}\right] = 0 \quad \text{or} \quad w_{2,x} = 0 \tag{25h}$$

$$cG_1\gamma_1 = 0 \quad \text{or} \quad w_1 + w_2 + \frac{c^2}{6F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right) = 0 \tag{25i}$$

As pointed out in the development of Appendix III, the thickness of the cylinder has been considered negligibly small with respect to the radius; thus, the compressive load is distributed evenly over the two faces.

In addition to the general instability type of buckling, the phenomenon of very short wave deformations (face wrinkling) receives special attention herein. The length of a cylinder is supposed to be large enough as compared to the wavelength of any typical deformation pattern to allow the effect of boundary conditions on the magnitude of the critical loads to be ignored. The set of nine boundary conditions (Equation (25)) gives rise to a very large number of possible combinations of generalized forces and displacements. However, although boundary conditions are of decisive importance for edge deformation, their importance decreases as the number of repeated deformation patterns along the length of the cylinder increases.

EQUIVALENT SINGLE GOVERNING EQUATION

It is convenient to consider a single partial-differential equation in one of the unknowns rather than the eight simultaneous equations in eight unknowns. In this way, the relative importance of the stiffness parameters and geometric quantities can be assessed, and the characteristic equation for the "classical" solution can be extracted directly; in addition, the specialization to previous, but less general, theories for plates and shells can be readily accomplished.

The governing equations can be written in the form

$$A\,\bar{r} = 0 \tag{26}$$

where $A$ is the matrix of linear operators and $\bar{r}$ is a column vector whose components are the functions $u_1$, $u_2$, $v_1$, $v_2$, $w_1$, $w_2$, $c\gamma_1$, and $c\gamma_2$.

The matrix $A$ is diagonalized when postmultiplied by its adjoint to yield

$$A\,\|A_{ij}\|' = |A|\,I \tag{27}$$

where $\|A_{ij}\|'$ is the adjoint of $A$ or the transposed matrix of the cofactors of the elements of $A$, and $I$ is the unit matrix. $|A|$ is the determinant of $A$. Instead of Equation (26), the equation

$$|A|\,I\,\bar{r} = 0 \tag{28}$$

may be solved; that is, a nontrivial solution to the equation, say, for the radial displacement $w_1$,

$$|A|\,w_1 = L(w_1) = 0 \tag{29}$$

must be found. The symmetric operator matrix $A$ is

$$
A = \begin{bmatrix}
C\left[(\,)_{,xx} + \frac{1-\nu}{2}(\,)_{,\varphi\varphi}\right] & 0 & C\frac{1+\nu}{2}(\,)_{,x\varphi} & 0 & \frac{\nu C}{R}(\,)_{,x} & 0 & -\frac{G_1}{c}(\,) & 0 \\
0 & C\left[(\,)_{,xx} + \frac{1-\nu}{2}(\,)_{,\varphi\varphi}\right] & 0 & C\frac{1+\nu}{2}(\,)_{,x\varphi} & 0 & \frac{\nu C}{R}(\,)_{,x} & \frac{G_1}{c}(\,) & 0 \\
C\frac{1+\nu}{2}(\,)_{,x\varphi} & 0 & C\left[(\,)_{,\varphi\varphi} + \frac{1-\nu}{2}(\,)_{,xx}\right] & 0 & \frac{C}{R}(\,)_{,\varphi} & 0 & 0 & -\frac{G_2}{c}(\,) \\
0 & C\frac{1+\nu}{2}(\,)_{,x\varphi} & 0 & C\left[(\,)_{,\varphi\varphi} + \frac{1-\nu}{2}(\,)_{,xx}\right] & 0 & \frac{C}{R}(\,)_{,\varphi} & 0 & \frac{G_2}{c}(\,) \\
\frac{\nu C}{R}(\,)_{,x} & 0 & \frac{C}{R}(\,)_{,\varphi} & 0 & D\nabla^4(\,) + \left[\frac{F}{c} + \frac{C}{R^2}\right](\,) + N_x(\,)_{,xx} & -\frac{F}{c}(\,) & -G_1\frac{c+t}{2c}(\,)_{,x} & -G_2\frac{c+t}{2c}(\,)_{,\varphi} \\
0 & \frac{\nu C}{R}(\,)_{,x} & 0 & \frac{C}{R}(\,)_{,\varphi} & -\frac{F}{c}(\,) & D\nabla^4(\,) + \left[\frac{F}{c} + \frac{C}{R^2}\right](\,) + N_x(\,)_{,xx} & -G_1\frac{c+t}{2c}(\,)_{,x} & -G_2\frac{c+t}{2c}(\,)_{,\varphi} \\
-\frac{G_1}{c}(\,) & \frac{G_1}{c}(\,) & 0 & 0 & -G_1\frac{c+t}{2c}(\,)_{,x} & -G_1\frac{c+t}{2c}(\,)_{,x} & \frac{G_1}{c}(\,) - \frac{c^2G_1^2}{12F}(\,)_{,xx} & -\frac{c^2G_1G_2}{12F}(\,)_{,x\varphi} \\
0 & 0 & -\frac{G_2}{c}(\,) & \frac{G_2}{c}(\,) & -G_2\frac{c+t}{2c}(\,)_{,\varphi} & -G_2\frac{c+t}{2c}(\,)_{,\varphi} & -\frac{c^2G_1G_2}{12F}(\,)_{,x\varphi} & \frac{G_2}{c}(\,) - \frac{c^2G_2^2}{12F}(\,)_{,\varphi\varphi}
\end{bmatrix}
\tag{30}
$$

The linear operator $L$ , corresponding to the determinant $|A|$ , is

$$
\begin{aligned}
L = {} & \left[ D\nabla^4(\;) + N_x(\;)_{,xx} + \left(\frac{Et}{R^2} + \frac{2F}{c}\right)(\;) \right] \\
& \cdot \left[ \frac{(c+t)^2}{2c} \nabla^8 \left\{ \frac{4(\;)}{Cc(1-\nu)} - \frac{1}{G_1}(\;)_{,\varphi\varphi} - \frac{1}{G_2}(\;)_{,xx} \right\} \right. \\
& + \left\{ D\nabla^8(\;) + N_x\nabla^4(\;)_{,xx} + \frac{Et}{R^2}(\;)_{,xxxx} \right\} \\
& \cdot \left\{ \frac{\nabla^4(\;)}{G_1G_2} - \frac{4}{Cc(1-\nu)} \left( \frac{1}{G_1}(\;)_{,xx} + \frac{1}{G_2}(\;)_{,\varphi\varphi} \right) \right. \\
& \left.\left. + \left( \frac{2}{Cc} + \frac{c^2\nabla^4}{12F} \right) \left( \frac{4(\;)}{Cc(1-\nu)} - \frac{1}{G_1}(\;)_{,\varphi\varphi} - \frac{1}{G_2}(\;)_{,xx} \right) \right\} \right] \\
& + \frac{4}{R^2c} \left[ D\nabla^8(\;) + N_x\nabla^4(\;)_{,xx} + \frac{Et}{R^2}(\;)_{,xxxx} \right] \\
& \cdot \left[ \frac{\nu^2}{2} \left\{ \frac{4(\;)}{Cc(1-\nu)} - \frac{1}{G_1}(\;)_{,\varphi\varphi} - \frac{1}{G_2}(\;)_{,xx} \right\} + 2\left\{ \frac{c^2}{12F}(\;)_{,xx} + \frac{1+\nu}{2G_2}(\;) \right\}_{,\varphi\varphi} \right. \\
& - \frac{c^2}{12F} \frac{Cc(1-\nu^2)}{4} \nabla^2 \left\{ \frac{4(\;)}{Cc(1-\nu)} - \frac{1}{G_1}(\;)_{,\varphi\varphi} - \frac{1}{G_2}(\;)_{,xx} + \frac{12F}{G_1G_2c^2}(\;) \right\}_{,\varphi\varphi} \\
& \left. - \frac{c^2}{12F} \frac{Cc(1-\nu^2)}{4} \left\{ \frac{4(\;)}{Cc(1-\nu)} - \frac{1}{G_1}(\;)_{,\varphi\varphi} - \frac{1}{G_2}(\;)_{,xx} + \frac{12F}{G_1G_2c^2}(\;) \right\}_{,xx\varphi\varphi} \right] \\
& + \frac{4(c+t)^2}{R^2c^2} \left[ \nabla^4(\;)_{,xx\varphi\varphi} - \frac{Cc(1-\nu)}{8} \nabla^4 \left\{ \frac{4(\;)}{Cc(1-\nu)} - \frac{1}{G_1}(\;)_{,\varphi\varphi} - \frac{1}{G_2}(\;)_{,xx} \right\}_{,xx\varphi\varphi} \right. \\
& \left. - \frac{Cc(1-\nu)}{8} \nabla^6 \left\{ \frac{4(\;)}{Cc(1-\nu)} - \frac{1}{G_1}(\;)_{,\varphi\varphi} - \frac{1}{G_2}(\;)_{,xx} \right\}_{,\varphi\varphi} \right] \qquad (31)
\end{aligned}
$$

The order of the highest derivative in this operator is 18; the number of independent conditions that can be imposed at the boundary, in terms of generalized forces, displacements, or combinations thereof, is 9 . Equation (31) relative to existing equations of thin plates and shell theory is discussed in a subsequent section.

It may be noted that the operator in the governing equation (Equation (29)) can be written as

$$L = L_0 + \left(\frac{c}{R}\right)^2 L_1 \tag{32}$$

As terms of the order $c/R$ have been neglected with respect to unity, the second term on the right-hand side of Equation (32) can be ignored if

$$\frac{c}{R} \frac{L_1 w}{L_0 w} = O(1)$$

An estimate of the relative importance of the last part of the operator L is made in conjunction with the analyses of the axisymmetric and non-axisymmetric buckling modes carried out in the following sections.

Axisymmetric Buckling

For the case of axisymmetric behavior, the deformations $v_1$ , $v_2$ , and $\gamma_2$ , as well as all derivatives with respect to $\varphi$ , vanish and the resulting governing operator becomes

$$L = \left[ \frac{4}{Cc(1-\nu)} - \frac{1}{G_2} {}_{,xx} \right]_{,xx} L_a$$

where

$$L_a = \left[ D(\ )_{,xxxx} + N_x(\ )_{,xx} + \left( \frac{Et}{R^2} + \frac{2F}{c} \right)(\ ) \right]_{,xx}$$

$$\cdot \left[ \frac{(c+t)^2}{2c} (\ )_{,xxxx} + \left\{ D(\ )_{,xxxx} + N_x(\ )_{,xx} + \frac{Et}{R^2} (\ ) \right\} \right.$$

$$\left. \cdot \left\{ \frac{c^2}{12F} (\ )_{,xxxx} - \frac{1}{G_1} (\ )_{,xx} + \frac{2}{Cc} \right\} \right]$$

$$+ \frac{2\nu^2}{R^2 c} \left[ D(\ )_{,xxxx} + N_x(\ )_{,xx} + \frac{Et}{R^2} \right]_{,xx} \tag{33}$$

Thus, for the case of axisymmetric buckling, the 18th order governing equation (Equation (29)) can be replaced by a 14th order equation. On the basis of the same considerations, the number of boundary conditions is reduced from 9 to 7. Specifically, the boundary conditions given by Equations (25c) and (25d) become trivial.

A solution to the governing equation $L_a(\bar{r}) = 0$, corresponding to "classical" simple support of the faceplates (vanishing radial edge displacement and edge moment), is

$$\bar{r} = \begin{bmatrix} u_1 \\ u_2 \\ w_1 \\ w_2 \\ c\gamma_1 \end{bmatrix} = \begin{bmatrix} U_1 \cos \frac{\pi x}{\lambda} \\ U_2 \cos \frac{\pi x}{\lambda} \\ W_1 \sin \frac{\pi x}{\lambda} \\ W_2 \sin \frac{\pi x}{\lambda} \\ c\Gamma_1 \cos \frac{\pi x}{\lambda} \end{bmatrix} \tag{34}$$

The forms of $u_1$, $u_2$, and $\gamma_1$ are implied by the relationships that exist between these quantities and the radial displacements $w_1$ and $w_2$

in the original governing equations. The "implied" boundary conditions for the case of "classical" simple support of the faces

$$w_1 = D\left[w_{1,xx} + \nu w_{1,\varphi\varphi}\right] = 0 \tag{35a}$$

$$w_2 = D\left[w_{2,xx} + \nu w_{2,\varphi\varphi}\right] = 0 \tag{35b}$$

are

$$C\left[w_{1,x} + \nu \frac{w_1}{R}\right] = 0 \tag{35c}$$

$$C\left[u_{2,x} + \nu \frac{w_2}{R}\right] = 0 \tag{35d}$$

$$w_1 + w_2 + \frac{c^2}{6F} G_1 \gamma_{1,x} = 0 \tag{35e}$$

With the use of the dimensionless parameters

$$k = \frac{N_x c^2}{\pi^2 D} \qquad \text{(load parameter)}$$

$$\beta = \frac{c}{\lambda} = \frac{c}{L} m \qquad \text{(wavelength parameter)}$$

$$\zeta = \frac{c^4}{t^2 R^2} \frac{12(1-\nu^2)}{\pi^4} \qquad \text{(curvature parameter)}$$

$$\epsilon = \frac{t}{c} \qquad \text{(thickness ratio parameter)}$$

and

$$\rho = \sqrt{\frac{G_2}{G_1}}$$

$$\theta = \frac{E}{\sqrt{G_1 G_2}} \left(\frac{t}{c}\right)^3 \frac{\pi^2}{2(1-\nu)^2} \qquad \text{(stiffness parameters)}$$

$$\varphi = \frac{F}{E} \left(\frac{c}{t}\right)^3 \frac{24(1-\nu^2)}{4}$$

a characteristic equation results as

$$\left[\beta^4 - k\beta^2 + \zeta + \varphi\right] \left[3(1+\epsilon)^2 \beta^4 + \left\{\beta^4 - k\beta^2 + \zeta\right\} \left\{\frac{\beta^4}{\varphi} + \theta\rho\beta^2 + \epsilon^2\right\}\right] + \frac{\zeta^2\epsilon^2\nu^2}{1-\nu^2} \left[\beta^4 - k\beta^2 + \zeta\right] = 0 \qquad (36)$$

Equation (36) is of the form

$$Ak^2 - Bk + C = 0 \qquad (37)$$

where $A$ , $B$ , and $C$ are polynomials in the wavelength parameter $\beta$ . As $A$ , $B$ , $C$ , and $B^2 - 4AC$ are positive definite quantities, two real-valued positive roots are obtained for each value of the wavelength corresponding to different values of the ratio $W_1/W_2$ . The relative importance of the last term of the left-hand side of Equation (36) can be readily determined by examining the coefficients $B$ and $C$ of Equation (37). With $N$ defined as $\zeta\epsilon^2\nu^2/(1-\nu^2)$ and irrelevant terms omitted, the expressions become

$$B(\beta^2) = \frac{2}{\varphi}\beta^{10} + \ldots + \left[\varphi\epsilon^2 \left\{1 + \frac{N}{\varphi\epsilon^2\nu^2}(2-\nu^2)\right\}\right]\beta^2$$

$$C(\beta^2) = \frac{1}{\varphi}\beta^{12} + \ldots + \left[N\frac{5-4\nu^2}{\nu^2} + \zeta\left\{3 + 6\epsilon + \frac{\zeta}{\varphi}\right\} + \varphi\left\{3 + 6\epsilon + 4\epsilon^2 + \frac{\zeta}{\varphi}\right\}\right]\beta^4$$

$$+ \ldots + \left[\zeta\varphi\epsilon^2\left\{1 + \frac{N}{\varphi\epsilon^2\nu^2}\right\}\right]$$

The quantity

$$\frac{N}{\varphi\epsilon^2\nu^2} \approx \frac{\zeta}{\varphi} = \frac{Et}{2FR}\left(\frac{c}{R}\right)$$

is of the order of magnitude $c/R$ for typically large values of $E/2F$ (for example, $2 \times 10^2$ ) and $t/R$ (for example, $5 \times 10^{-3}$ ) and is therefore to be neglected with respect to unity. Hence, the contribution of $N$ to the last term in both $B(\beta^2)$ and $C(\beta^2)$ is negligible. The effect of $N$ in the coefficient $\beta^4$ in $C(\beta^2)$ is certainly small, as

$$\frac{N}{\varphi\nu^2} \leq O\left(\epsilon^2\frac{c}{R}\right) \ll O\left(\frac{c}{R}\right)$$

These and other simplifications of the same nature, although not essential for a numerical computation of critical values of the load and the wavelength, allow an analytical evaluation of both in closed form. When the N-term is negligible, the characteristic equation simplifies to

$$\left[\beta^2 k - (\beta^4 + \varphi + \zeta)\right] \cdot \left[\left\{\frac{\beta^6}{\varphi} + \theta\rho\beta^4 + \epsilon^2\beta^2\right\}k\right.$$
$$\left. - \left\{3(1+\epsilon)^2\beta^4 + \left(\frac{\beta^4}{\varphi} + \theta\rho\beta^2 + \epsilon^2\right)(\beta^4 + \zeta)\right\}\right] = 0 \qquad (38)$$

The characteristic equation (Equation (36)) and its simplified form (Equation (38)) are solved for the load parameter $k$ by using typical values of the curvature, relative thickness, and stiffness parameters.[10]

The load parameter $k$ versus the wavelength parameter $\beta$, considered to be a continuous variable, is plotted in Figure 4. In Figure 5, a similar solution is presented for values of the various parameters corresponding to low values of $F$ and $G_1$ to assess qualitatively the buckling of a relatively weak-core sandwich shell. For comparison, corresponding solutions based on existing theories (References 11 and 8) are shown also. No quantitative distinction can be made between the solutions of the complete and simplified quadratic characteristic equations; thus, only a single curve is evidenced for each branch of the solution.

The characteristic equation is of the form

$$(k - k_1)\,(k - k_2) = 0 \tag{39}$$

where

$$k_1(\beta^2) = \beta^2 + \frac{\varphi + \zeta}{\beta^2} \tag{40}$$

and

$$k_2(\beta^2) = \frac{3(1 + \epsilon)^2\,\beta^2}{\frac{\beta^4}{\varphi} + \theta\rho\beta^2 + \epsilon^2} + \beta^2 + \frac{\zeta}{\beta^2} \tag{41}$$

Stationary values of $k_1$ or $k_2$ occur whenever $\partial k/\partial(\beta^2) = 0$, and a critical situation is reached when the stationary value of $k$ is a minimum $(\partial^2 k/\partial(\beta^2)^2 > 0)$. As $\beta$ is a real-valued quantity, $\beta^2$ is always nonnegative and, in practice, always positive. The root $k_1$ has only one stationary value; a minimum occurs when

$$\left(\beta_{cr}\right)_1 = (\varphi + \zeta)^{1/4} \simeq \varphi^{1/4} \tag{42}$$

and the corresponding critical load is given by

$$\left(k_{cr}\right)_1 = 2\sqrt{\varphi + \zeta} \simeq 2\sqrt{\varphi} \tag{43}$$

since, as mentioned earlier, $\zeta/\varphi$ may be neglected in comparison with unity. The root $k_2$ usually has more than one stationary value in the $\beta$ interval under consideration. This may be seen in Figures 4 and 5. For the more practical set of core parameters, minimums occur, corresponding to two very distinct values of $\beta$.

The longwave instability mode $(\beta \ll 1)$ can be analyzed by neglecting terms which are obviously small due to the small size of $\beta$. Equation (41) then simplifies to

$$k_2 \simeq \frac{3(1+\epsilon)^2 \beta^2}{\theta\rho\beta^2 + \epsilon^2} + \frac{\zeta}{\beta^2} \tag{44}$$

From this expression, a critical wavelength parameter

$$\left(\beta_{cr}\right)_{cl} = \epsilon\zeta^{1/2}\left[\epsilon(1+\epsilon)\sqrt{3\zeta} - \theta\rho\zeta\right]^{-1/2} \tag{45}$$

and the corresponding critical load parameter

$$\left(k_{cr}\right)_{cl} = 2(1+\epsilon)\sqrt{3\zeta/\epsilon^2} - \theta\rho\zeta/\epsilon^2 \tag{46}$$

are determined. It may be noted that the terms neglected in Equation (44) are those originating from the inclusion of the effects of face bending stiffness and the core normal strain. Obviously, the result obtained corresponds to the classical solution of Reference 11; therefore, the critical wavelength and load have been assigned a subscript "cl" for this case.

For the shortwave instability mode, $\beta$ is of the order of magnitude of unity or even larger. Equation (41) may then be simplified to

$$k_2 \simeq \frac{3(1+\epsilon)^2}{\beta^2/\varphi + \theta\rho} + \beta^2 \tag{47}$$

The critical wavelength and associated load parameter are

$$\left(\beta_{cr}\right)_2 = \left[(1+\epsilon)\sqrt{3\varphi} - \theta\rho\varphi\right]^{1/2} \tag{48}$$

and

$$\left(k_{cr}\right)_2 = 2(1+\epsilon)\sqrt{3\varphi} - \theta\rho\varphi \tag{49}$$

The results presented in this section are discussed in detail in the subsequent section entitled "RESULTS AND DISCUSSION".

Nonaxisymmetric Buckling

For the case of nonaxisymmetric buckling, a solution of the general equation (Equation (28)) is sought. Such a solution, corresponding to "classical" simple support of the faceplates (vanishing edge displacement and edge bending moment), is available in the form

$$\bar{r} = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ w_1 \\ w_2 \\ c\gamma_1 \\ c\gamma_2 \end{bmatrix} = \begin{bmatrix} U_1 \cos \pi x/\lambda \sin n\varphi \\ U_2 \cos \pi x/\lambda \sin n\varphi \\ V_1 \sin \pi x/\lambda \cos n\varphi \\ V_2 \sin \pi x/\lambda \cos n\varphi \\ W_1 \sin \pi x/\lambda \sin n\varphi \\ W_2 \sin \pi x/\lambda \sin n\varphi \\ c\Gamma_1 \cos \pi x/\lambda \sin n\varphi \\ c\Gamma_2 \sin \pi x/\lambda \cos n\varphi \end{bmatrix} \tag{50}$$

The forms of $u_1$ , $u_2$ , $v_1$ , $v_2$ , $\gamma_1$ , and $\gamma_2$ are implied by relationships that exist between these quantities and the radial displacements $w_1$ and $w_2$ in the original governing equations. The "implied" boundary conditions for the case of "classical" simple support of the faceplates

$$w_1 = D\left[w_{1,xx} + \nu w_{1,\varphi\varphi}\right] = 0 \tag{51a}$$

$$w_2 = D\left[w_{2,xx} + \nu w_{1,\varphi\varphi}\right] = 0 \tag{51b}$$

are

$$C\left[u_{1,x} + \nu \frac{w_1}{R}\right] = 0 \tag{51c}$$

$$v_1 = 0 \tag{51d}$$

$$C\left[u_{2,x} + \nu \frac{w_2}{R}\right] = 0 \tag{51e}$$

$$v_2 = 0 \tag{51f}$$

$$w_1 + w_2 + \frac{c^2}{6F} G_1\gamma_{1,x} = 0 \tag{51g}$$

With the use of dimensicnless parameters previously defined and the buckle aspect ratio parameter

$$p = \frac{n^2\lambda^2}{\pi^2 R^2} = \left(\frac{\lambda_x}{\lambda_\varphi}\right)^2$$

a characteristic equation is obtained as

$$
\begin{aligned}
&\left[\beta^4(1+p)^2 - k\beta^2 + \zeta + \varphi\right] \cdot \Bigg[3(1+\epsilon)^2\,\beta^8(1+p)^4 \left\{\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\,\beta^2(1+p\rho^2)\right\} \\
&\quad + \left[\beta^4(1+p)^4 - k\beta^2(1+p)^2 + \zeta\right] \\
&\quad \cdot \Bigg[\theta\rho(1+p)\,\beta^6\left\{\frac{2\epsilon^2}{1-\nu}\cdot\frac{1+p/\rho^2}{1+p} + \frac{\theta}{\rho}\,\beta^2(1+p)\right\} \\
&\quad + \left\{\epsilon^2\beta^4 + \frac{\beta^8}{\varphi}(1+p)^2\right\}\cdot\left\{\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\,\beta^2(1+p\rho^2)\right\}\Bigg]\Bigg] \\
&\quad + \Bigg\{\zeta\beta^4\left[\beta^4(1+p)^4 - k\beta^2(1+p)^2 + \zeta\right] \\
&\quad \cdot \Bigg[\frac{\nu^2\epsilon^2}{1-\nu^2}\left\{\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\,\beta^2(1+p\rho^2)\right\} - \frac{2\epsilon^2}{1-\nu^2}\,\frac{\theta}{\rho}\,p\beta^2\left(1+\nu+\frac{2\rho}{\theta\varphi}\,\beta^2\right) \\
&\quad - \frac{\beta^4}{\varphi}\,p(2+p)\left\{\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\,\beta^2(1+p\rho)^2 + \theta^2\varphi\right\}\Bigg] \\
&\quad + \zeta\beta^8 p(1+p)^2\,(1+\epsilon)^2 \\
&\quad \cdot \left[\frac{12\epsilon^2}{1-\nu^2} - \frac{3}{1+\nu}\,(p+2)\left\{\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\,\beta^2(1+p\rho^2)\right\}\right]\Bigg\}^{*} = 0 \qquad (52)
\end{aligned}
$$

This equation reduces to the characteristic equation for the case of axisymmetric buckling (Equation (36)) when the wave aspect ratio p is set equal to zero. For nonaxisymmetric buckling, p is positive definite and the last part of the characteristic equation, marked by an asterisk, can be ignored if the condition

$$\zeta/\varphi \leq O(c/R) \tag{53}$$

is satisfied.

Then, Equation (52) reduces to the form

$$(k - k_1)\ (k - k_2) = 0 \tag{54}$$

where

$$k_1 = \beta^2(1 + p)^2 + \frac{\zeta + \varphi}{\beta^2} \tag{55}$$

and

$$k_2 = \beta^2(1+p)^2 + \frac{\zeta}{\beta^2(1+p)^2}$$

$$+ \frac{3(1+\epsilon)^2\,\beta^6(1+p)^2\left[\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\,\beta^2(1+p\rho^2)\right]}{\theta\rho(1+p)\beta^6\left[\frac{2\epsilon^2}{1-\nu}\cdot\frac{1+p/\rho^2}{1+p} + \frac{\theta}{\rho}\beta^2(1+p)\right] + \left[\epsilon^2\beta^4 + \frac{\beta^8}{\varphi}(1+p)^2\right]\left[\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\beta^2(1+p\rho^2)\right]} \tag{56}$$

The first factor on the left-hand side of Equation (54) leads to the critical value of the wave parameter

$$\left(\beta_{cr}\right)_1 = \frac{(\varphi + \zeta)^{1/4}}{\sqrt{1 + p}} = \frac{1}{\sqrt{1 + p}} \cdot \left\{\left(\beta_{cr}\right)_1\right\}_{p=0} \tag{57}$$

and the associated critical load parameter

$$\left(k_{cr}\right)_1 = 2(1 + p)\sqrt{\varphi + \zeta} \simeq 2(1 + p)\sqrt{\varphi} = (1 + p)\left\{\left(k_{cr}\right)_1\right\}_{p=0} \tag{58}$$

Since $p$ is positive definite, the critical wavelength and the critical load are larger than in the case of axisymmetric buckling $(p = 0)$.

The second root of Equation (54), $k_2$, is rewritten by dividing the last terms by the nonzero factor

$$\frac{2\epsilon^2}{1-\nu} + \frac{\theta}{\rho}\,\beta^2(1 + p\rho^2)$$

Then,

$$k_2 = \beta^2(1+p)^2 + \frac{\zeta}{\beta^2(1+p)^2} + \frac{3(1+\epsilon)^2\,\beta^2(1+p)^2}{\frac{\beta^4}{\varphi}(1+p)^2 + f\theta\rho\beta^2(1+p) + \epsilon^2} \tag{59}$$

where

$$f = \frac{\frac{\theta}{\rho}\,\beta^2(1+p) + \frac{2\epsilon^2}{1-\nu}\cdot\frac{1+p/\rho^2}{1+p}}{\frac{\theta}{\rho}\,\beta^2(1+p\rho^2) + \frac{2\epsilon^2}{1-\nu}} \tag{60}$$

The factor $f$ depends on the ratio of the core shear moduli $\rho^2$ and is equal to unity for the case of an isotropic core $(\rho^2 = 1)$. It can be observed that the dependence of $f$ on $\beta^2$ is not large, since $\rho = \sqrt{G_2/G_1}$ possesses values in the interval 0.6 to 1.5 for practical honeycomb-core materials. The wave aspect ratio parameter $p = (\lambda_x/\lambda_\varphi)^2$ is less than or about 0.25 for longwave deformations.

In particular, the ratio

$$\frac{\beta^2\,\frac{\partial f}{\partial\beta^2}}{f} = \frac{\frac{2\epsilon^2}{1-\nu}\,\frac{\theta}{\rho}\,\beta^2\,\frac{p}{1+p}\,(2-\rho^2-1/\rho^2)}{\left[\frac{\theta}{\rho}\,\beta^2(1+p\rho^2) + \frac{2\epsilon^2}{1-\nu}\right]\left[\frac{\theta}{\rho}\,\beta^2(1+p) + \frac{2\epsilon^2}{1-\nu}\,\frac{1+p/\rho^2}{1+p}\right]} \qquad \text{(Cont'd)}$$

$$= \frac{p(2-\rho^2-1/\rho^2)}{\left[1+p\rho^2 + \frac{2\epsilon^2\rho}{\theta\beta^2(1-\nu)}\right]\left[1+p/\rho^2 + \frac{\theta\beta^2(1-\nu)}{2\epsilon^2\rho}(1+p)^2\right]}$$

$$< \frac{p(2-\rho^2-1/\rho^2)}{\left(1 + \frac{2\epsilon^2\rho}{\theta\beta^2(1-\nu)}\right)\left(1 + \frac{\theta\beta^2(1-\nu)}{2\epsilon^2\rho}\right)} < \frac{p}{4}(2-\rho^2-1/\rho^2) \qquad (61)$$

is small with respect to unity. If $\rho = 1$ (isotropic core), then the ratio vanishes. For example, when

$$\frac{2\epsilon^2\rho}{\theta\beta^2(1-\nu)} = 10^{\pm 1}$$

then

$$\frac{\beta^2 \partial f/\partial\beta^2}{f} < \frac{p}{12}(2-\rho^2-1/\rho^2)$$

For very short wavelengths,

$$f \rightarrow \frac{1 + p}{1 + p\rho^2} \quad \text{and} \quad \frac{\partial f}{\partial\beta^2} \rightarrow 0$$

Upon neglect of the dependence of $f$ on $\beta^2$ and with the assumption that $\beta^2$ is small compared to unity (longwave buckling), the critical values for the wavelength and load parameters are obtained as

$$\left(\beta_{cr}\right)_{cl} = \frac{\epsilon\zeta^{1/2}}{\sqrt{1+p}}\left[\epsilon(1+\epsilon)\,(1+p)\,\sqrt{3\zeta} - f\theta\rho\zeta\right]^{-1/2} \qquad (62)$$

and

$$\left(k_{cr}\right)_{cl} = 2(1+\epsilon)\sqrt{3\zeta/\epsilon^2} - \frac{f}{1+p}\,\theta\rho\zeta/\epsilon^2 \tag{63}$$

for a given value of $p$. Equation (63) shows that when the core shear stiffness approaches infinity $(\theta \to 0)$, then

$$\left(k_{cr}\right)_{cl} = 2(1+\epsilon)\sqrt{3\zeta/\epsilon^2}$$

This result corresponds to nonaxisymmetric classical buckling of a homogeneous isotropic shell and is identical to the result obtained for the axisymmetric mode. However, when the shear stiffness is finite, Equations (63) and (46) (axisymmetric buckling case) indicate that the two classical sandwich shell results are different. Indeed, for practical values of the core shear stiffness parameter $(0.0 < \rho < 1.5)$, the nonaxisymmetric buckling mode leads to a slightly different value of the buckling load. The applicability of these equations for long shells to shells of finite length is discussed later.

For a shortwave buckling mode $(\beta = O(1))$, the critical wavelength parameter

$$\left(\beta_{cr}\right)_2 = \frac{\left[(1+\epsilon)\sqrt{3\varphi} - f\theta\rho\varphi\right]^{1/2}}{\sqrt{1+p}} \tag{64}$$

and the corresponding critical load parameter

$$\begin{aligned}\left(k_{cr}\right)_2 &= (1+p)\left[2(1+\epsilon)\sqrt{3\varphi} - f\theta\rho\varphi\right] \\ &= (1+p)\left[2(1+\epsilon)\sqrt{3\varphi} - \frac{1+p}{1+p\rho^2}\,\theta\rho\varphi\right]\end{aligned} \tag{65}$$

since for very short wavelengths,

$$f \rightarrow \frac{1+p}{1+p\rho^2}$$

Equation (65), when compared with Equation (49) for the axisymmetric case, indicates a higher value of the critical load parameter when

$$\frac{(1+p)^2}{1+p\rho^2} < 1 + \frac{2(1+\epsilon)\sqrt{3\varphi}}{\theta\rho\varphi} \tag{66}$$

In most cases this condition will be met; a more complete discussion follows in the section entitled "RESULTS AND DISCUSSION".

## RESULTS AND DISCUSSION

### THEORY

The general theory developed in this investigation is given by Equations (17), (18), (19), and (20). These relations represent the governing equations and associated boundary conditions which can be used to describe the bending, buckling, and postbuckling behavior of sandwich shells (and curved plates). This theory is applicable to sandwich shells which are constructed of thin, isotropic face layers and a honeycomb-type core layer; it is considered to be the most complete and consistent theory developed to date for such a configuration.

Of the many sandwich shell theories which exist in the literature, the recent studies (References 5 and 6) by Wempner et al. are the most advanced studies with which the present development can be compared. It would appear from a cursory examination that the two theories essentially describe the same problem; nevertheless, it is evident, as will be disclosed, that there are marked and important differences relative to both development and applicability.

A direct comparison with the work of Wempner et al. reveals that:

1. The treatment of the face layers is equivalent.
2. The core considerations utilized in Reference 5 are based on a model in which a contradiction exists between the linearized equilibrium equation employed and the nonlinear strain-displacement relation assumed. That is, the core is assumed to be in a linear state of antiplane stress while the kinematic relations tacitly imply a more complex behavior.
3. Even though both theories take into account the effect of core thickness with respect to radii of curvature, the theory of Reference 5 does not account for the fact that a honeycomb core (which is within the weak core family for which the theory is stated to be valid) does not possess the same core shell area at the the two interfaces. For

thin sandwich shells ($c/\rho$ negligible), this consideration is not important. However, for thick-core sandwich shells, the only example which is illustrated in Reference 5, the consideration is important and renders the results questionably applicable to a honeycomb-core sandwich or, in fact, to any thick-core sandwich in which an initially flat core medium is formed into a curved surface.

4. In Reference 6, which specifically treats thin sandwich shells, the final equations presented are inapplicable to the complete study of the buckling of such shells, since a gross assumption is invoked that precludes the occurrence of the antisymmetric face-wrinkling mode of instability.

5. The boundary conditions presented by Wempner et al. are developed independently of the governing equations and are stipulated in terms of edge forces only. The present theory develops both governing equations and force-displacement boundary conditions within the consistent framework of a complete variational treatment.

The governing equations (Equations (17), (18), and (19)) of the present theory can be shown to be equivalent to a single 18th-order partial differential equation in any one of the variables (for example, the normal displacement of a face layer). The associated boundary conditions, given by Equation (20), are consistent with the 18th-order system since 9 boundary conditions on either generalized force or displacement can be prescribed at an edge of the shell. A special case of the general equations is applicable to the buckling of a thin, circular cylindrical shell (see Equation (31)). This equation has been obtained by systematically reducing the eight governing equations for the thin, circular cylindrical sandwich shell in axial compression (Equation (24)) through the use of linear operators.

Before undertaking the discussion of the results of the application of the present theory to the compressive buckling of thin, circular cylindrical shells, it is of interest to show the relationship of the present single-governing equation of the 18th order with previous equations utilized to study the buckling of flat plates and circularly curved plates and cylinders, including the associated requirement for sufficient boundary conditions to effect complete solutions. With reference to Table I, it can be seen that the present 18th-order equation, requiring 9 boundary conditions per edge, reduces to the 7 other cases indicated when the prescribed simplifying assumptions are enforced. From the table, it is noted dramatically that the trend toward higher performance structures evidences increasing complexity with regard to theoretical considerations. This point, although the result of theoretical considerations, cannot be casually dismissed by individuals concerned with the design, development, analysis, and testing of aerospace components which reflect a strong structures-materials interface aimed at achieving increased structural efficiency.

AXISYMMETRIC BUCKLING OF THIN, CIRCULAR CYLINDRICAL SANDWICH SHELLS

The solution of Equation (29) for the case of axisymmetric buckling leads to the buckling criterion, Equation (36), which is presented graphically in Figure 4. The figure shows the variation of the buckling load parameter $k$ with wavelength for the cylinder described by the parameter values noted. As mentioned earlier, these parameter values reflect an actual cylinder, the test results for which are described in Reference 10. Figure 4 reveals that three distinct minimums occur for practical values of the core stiffness parameters. Each minimum corresponds to a different buckling mode; namely, one relatively long-wave antisymmetric mode and two relatively short-wave modes corresponding to antisymmetric and symmetric buckling. When the case of a shell with very low values of the core moduli is considered, a different behavior, leading to consonance of the two usual antisymmetric modes, is evidenced (Figure 5). Although, on the basis of weight considerations, such a structure might appear to be advantageous, the low shear stiffness and the correspondingly large shear deformations would render the composite impractical.

TABLE I

COMPARISON OF PLATE AND SHELL THEORIES WITH RESPECT TO ORDER OF GOVERNING PARTIAL DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS REQUIRED PER EDGE

| Case | Order of Governing Partial Differential Equation | Number of Boundary Conditions To Be Prescribed at an Edge |
|---|---|---|
| | **SHELLS** | |
| 1. Present theory | 18 | 9 |
| 2. Fulton (Ref. 8) $F \to \infty$ | 12 | 6 |
| 3. Stein-Mayers (Ref. 9) $F \to \infty$., $\epsilon \to 0$ | 10 | 5 |
| 4. Donnell (Ref. 12) $F \to \infty$, $G \to \infty$ | 8 | 4 |
| | **PLATES** | |
| 5. Benson-Mayers (Ref. 2) $R \to \infty$ | | |
| antisymmetric behavior | 10 | 5 |
| symmetric behavior | 4 | 2 |
| coupled behavior (e.g., dissimilar facings) | 18 | 9 |

TABLE I (Continued)

| Case | Order of Governing Partial Differential Equation | Number of Boundary Conditions To Be Prescribed at an Edge |
|---|---|---|
| **PLATES** | | |
| 6. Hoff (Ref. 13)<br>$R \to \infty$ , $F \to \infty$ | 8 | 4 |
| 7. Libove-Batdorf (Ref. 14)<br>$R \to \infty$ , $F \to \infty$ , $\epsilon \to 0$ | 6 | 3 |
| 7a. Reissner (Ref. 15)<br>Seide-Stowell (Ref. 16)<br>simply supported isotropic case only* | 4 | 2 |
| 8. Saint-Venant (Ref. 17)<br>$R \to \infty$ , $F \to \infty$ , $G \to \infty$ | 4 | 2 |

*In the case of simple support and an isotropic core, it is possible to factor a second-order operator from the governing equation. No solution other than the trivial one corresponds to this second-order equation. However, the fact that the fourth-order partial differential equation is, in general, a sixth-order equation has not been recognized to date, except in Reference 2.

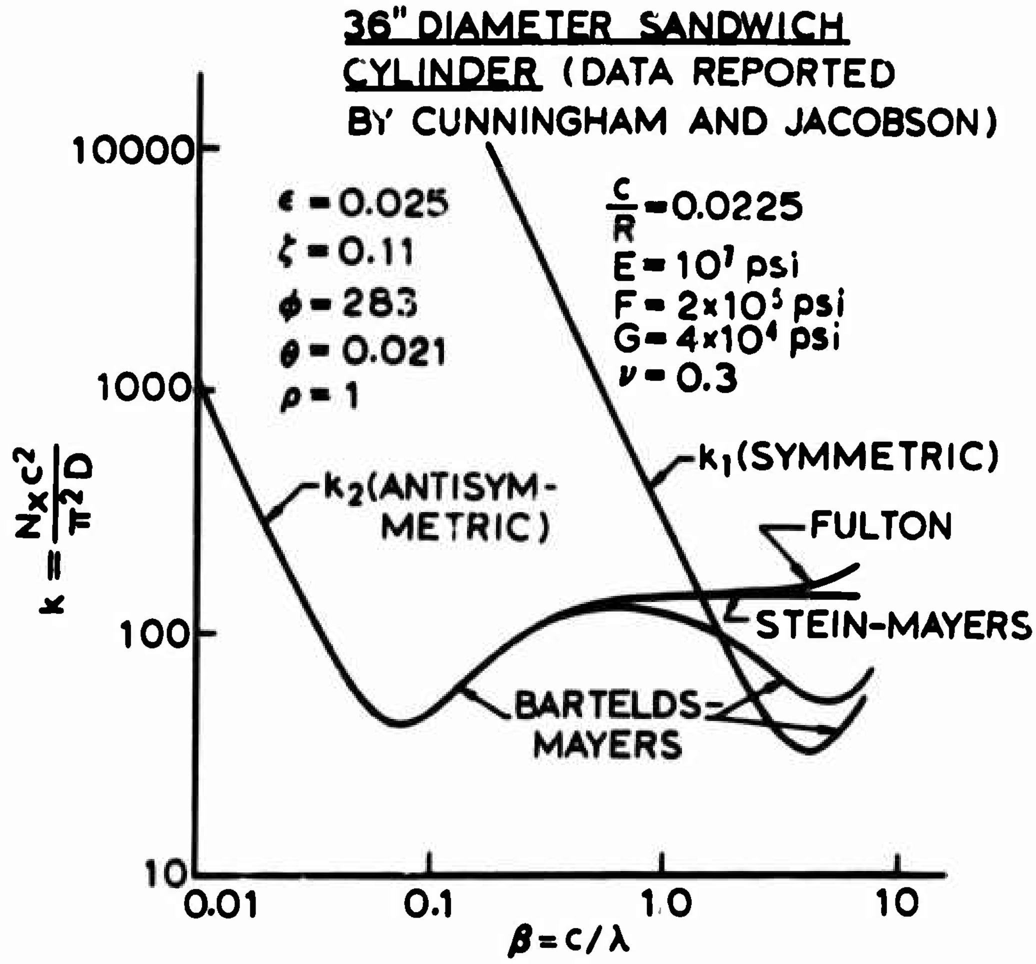


Figure 4. Relation Between Load Parameter k and Wavelength Parameter β for a Sandwich Cylinder of Practical Properties.

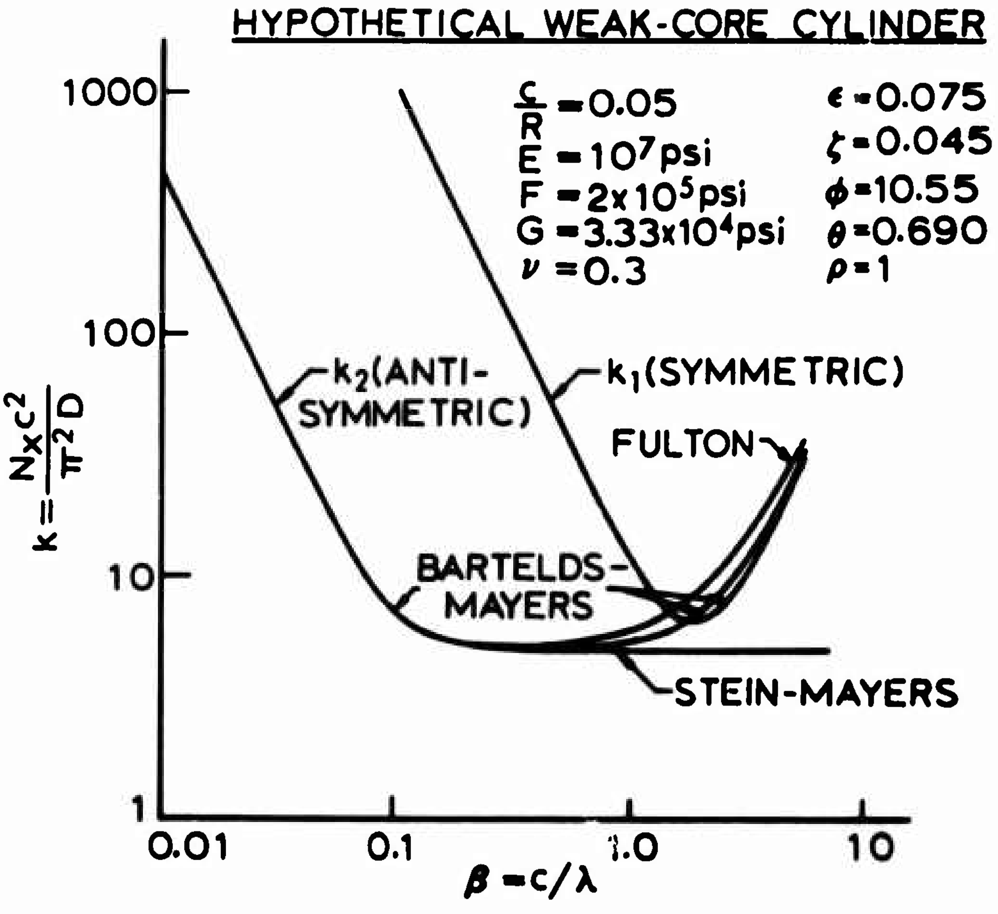


Figure 5. Relationship Between Load Parameter k and Wavelength Parameter $\beta$ for a Hypothetical Weak-Core Sandwich Cylinder.

The character of the different buckling modes (Figure 4) can be readily recognized if a deformation vector

$$\bar{x} = \begin{bmatrix} \bar{u} \\ \tilde{u} \\ \tilde{w} \\ \bar{w} \\ c\bar{\gamma} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ w_1 + w_2 \\ w_1 - w_2 \\ 2c\,\gamma_1 \end{bmatrix} \tag{67}$$

is considered. By performance of elementary row operations on the matrix A (Equation (26)), and after substitution of the solution given by Equation (34), this vector can be written as

$$\bar{x} = \begin{bmatrix} \beta^4 - k\beta^2 + \zeta + \varphi \\ \dfrac{(1-\nu^2)^{1/2}}{\nu} \; \dfrac{c}{(1+c)\,\beta^2 \;\; \overline{3\zeta}} \left(\beta^4 - k\beta^2 + \zeta\right)\left(\beta^4 - k\beta^2 + \varphi + \dfrac{\zeta}{1-\nu^2}\right) \\ \dfrac{6(1-\nu^2)^{1/2}}{\pi\nu} \; \dfrac{\beta}{c \;\; \overline{3\zeta}} \left(\beta^4 - k\beta^2 + \zeta + \varphi\right) \\ \dfrac{2}{\pi} \; \dfrac{1}{\beta(1+c)} \left(\beta^4 - k\beta^2 + \zeta\right) \\ - \dfrac{2(1-\nu^2)^{1/2}}{\nu} \; \dfrac{\theta\rho}{c(1+c) \;\; \overline{3\zeta}} \left(\beta^4 - k\beta^2 + \zeta\right)\left(\beta^4 - k\beta^2 + \zeta + \varphi\right) \end{bmatrix} \tag{68}$$

For a solution to Equation (40), only the components $\tilde{u}$ and $\bar{w}$ are nonzero. Since the ratio

$$\frac{\tilde{u}}{\bar{w}} = \frac{\pi \nu}{\sqrt{12(1-\nu^2)}} \; \frac{c\,\bar{\xi}}{\beta} \simeq \frac{\nu}{\pi\beta} \cdot \frac{c}{R} \tag{69}$$

and, usually, since $\beta > 1$ , the relative face displacement

$$\bar{w} = \left( \frac{w_1 - w_2}{2} \right)$$

is seen to be the significant deformation. Thus, the buckling mode is characterized by essentially symmetric displacements of the faces. The wavelength of this instability mode is of the order of, or smaller than, the core thickness (Equation (42)); this mode is referred to as the symmetric wrinkling mode.

On the basis of a comparison similar to that made in Equation (69), the solutions corresponding to Equations (44) and (47) are recognized as being essentially antisymmetric in nature $(\tilde{w} \gg \bar{w} \, , \, \tilde{u} \gg \bar{u})$ . These characteristics are more pronounced in the case of the longwave buckling mode than in the wrinkling mode. The former has already been related to the classical instability mode (Equation (46)); the latter is referred to as the antisymmetric wrinkling mode.

In Figures 4 and 5, the curves for the antisymmetric solution based on the classical theory of Reference 9 approaches an asymptotic value for large values of the wavelength parameter. Equation (44), which is valid also if $F \to \infty$ $(\varphi \to \infty)$ , reveals that for increasing $\beta$ , the load parameter $k_2$ approaches the classical theory value $3/\theta\rho$ if $\epsilon = 0$ (membrane faces). The corresponding value of the load

$$2N_x = G_1 c \tag{70}$$

is referred to as the core shear-failure load in the classical theory. When the bending stiffness of the faces is included (Reference 8), the value of the load parameter does not tend to the asymptotic value but increases rapidly with increasing values of the wavelength parameter. This case is described by Equation (47) of the present analysis when $\varphi \rightarrow \infty$ . The solution based on the theory of Reference 5, which employs the same degree of freedom as the present development, does not reveal the existence of the antisymmetric wrinkling mode for reasons explained earlier. That is, an arbitrary, and obviously eminent, assumption that the effect of normal strain is of negligible order in antisymmetric deformation of the faces precludes the occurrence of a possible buckling mode.

Closed-form expressions for the critical wavelength parameters, the associated load parameters, and the stresses are shown in Table II for each of the three instability modes discussed. It can be concluded from these results that the symmetric wrinkling mode, for practical parameter values, corresponds to a lower value of the load parameter than the antisymmetric wrinkling mode.* If the converse were to be true, then

$$2(1 + \epsilon) \sqrt{3\varphi} - \theta\rho\varphi < 2 \sqrt{\varphi}$$

or

$$\frac{\theta\rho\varphi}{2(1 + \epsilon) \sqrt{3} - 2} > \sqrt{\varphi} \tag{71}$$

However, for the left-hand side of the inequality, the assumption

$$\beta^2 = (1 + \epsilon) \sqrt{3\varphi} - \theta\rho\varphi > 1$$

*This same conclusion is evidenced in the behavior of axially compressed, simply supported columns and plates.

TABLE II

AXISYMMETRIC INSTABILITY MODES

| | Classical Instability | Antisymmetric Wrinkling | Symmetric Wrinkling |
|---|---|---|---|
| Critical wavelength parameter $\beta$ : | $\epsilon\zeta^{1/2}\left[\epsilon(1+\epsilon)\sqrt{3\zeta} - \theta\rho\zeta\right]^{-1/2}$ (Eq. (45)) | $\left[(1+\epsilon)\sqrt{3\varphi} - \theta\rho\varphi\right]^{1/2}$ (Eq. (48)) | $\varphi^{1/4}$ (Eq. (42)) |
| Critical load parameter k : | $2(1+\epsilon)\sqrt{3\zeta/\epsilon^2} - \theta\rho\zeta/\epsilon^2$ (Eq. (46)) | $2(1+\epsilon)\sqrt{3\varphi} - \theta\rho\varphi$ (Eq. (49)) | $2\sqrt{\varphi}$ (Eq. (43)) |
| Critical stress $\sigma_x$ : | $\frac{c+t}{c}\frac{Ec}{R\sqrt{1-\nu^2}}\left[1 - \frac{1}{2G_1(c+t)} \cdot \frac{Ec}{R\sqrt{1-\nu^2}}\right]$ | $\frac{c+t}{c}\sqrt{\frac{2EFt}{c(1-\nu^2)}}\left[1 - \frac{t}{2G_1(c+t)} \cdot \sqrt{\frac{2EFt}{c\sqrt{1-\nu^2}}}\right]$ | $\sqrt{\frac{2EFt}{3c\sqrt{1-\nu^2}}}$ |
| Assumption: | $\beta \ll 1$ | $\beta > 1$ | $\zeta \ll \varphi$ |

is made; that is,

$$\overline{\varphi} > \frac{1 + \theta\rho\varphi}{(1 + \epsilon)\sqrt{3}} \tag{72}$$

The quantity $\theta\rho\varphi = (12/\pi^2)(F/G_1)$ reflects values from 5 to 10 for practical honeycomb-core materials. If, for example, $\epsilon = 0.05$, then the conditions Equations (71) and (72) require that

$$1 + \theta\rho\varphi < 1.82 \quad \overline{\varphi} \quad < 1.11\ \theta\rho\varphi \tag{73}$$

and the interval $[1 + \theta\rho\varphi\ ,\ 1.1\ \theta\rho\varphi]$ is nonzero only if $\theta\rho\varphi > 9$ ; this appears to be highly unusual. Thus, the critical instability mode is either the classical mode or the symmetric wrinkling mode, according to the present theory. It is also observed that both wrinkling modes show wavelengths that are smaller than the thickness of the shell and possibly of the order of magnitude of the cell size of the core material. The extreme situation requires a treatment of the core as an aggregate consisting of discrete elements rather than an idealized continuum.

As long as the theory is applicable, any intent to design a structure with the same safety margin with respect to different instability modes requires that

$$2(1 + \epsilon)\sqrt{3\zeta/\epsilon^2 - \theta\rho\zeta/\epsilon^2} = 2\ \overline{\varphi} \tag{74}$$

or, approximately $(\epsilon \ll 1)$ ,

$$\theta\rho\varphi = -2\frac{\epsilon^2}{\zeta}\varphi^{3/2} + 2\frac{3\epsilon^2}{\zeta}\varphi \tag{75}$$

For each assumed value of the curvature parameter $\zeta$ , the relation between $\theta\rho\varphi$ , which is proportional to $F/G_1$ (a characteristic quantity for a core material) and the core stiffness parameter $\varphi$ can be determined. Figure 6 shows a stability boundary for the two modes of buckling. The ranges of practical values of $\theta\rho\varphi$ are indicated for two typical core materials.

## NONAXISYMMETRIC BUCKLING OF THIN, CIRCULAR CYLINDRICAL SANDWICH SHELLS

The solution to Equation (29) for the case of nonaxisymmetric buckling leads to the buckling criterion given by Equation (52). The critical values of the wavelength parameter $\beta$ and the corresponding load parameter $k$, obtained from this equation, are shown in Table III. A comparison with the corresponding results for the case of axisymmetric buckling discloses that for both wrinkling modes, the critical values of the wavelength parameter are $1/\sqrt{1 + p}$ times the values for the axisymmetric case. In the isotropic case $(\rho = 1)$, the critical load parameter is $(1 + p)$ times as large for both wrinkling modes.

The longwave buckling mode (classical mode) also reveals a higher buckling load for the nonaxisymmetric case, as the positive factor

$$\frac{f}{1 + p} = \frac{\dfrac{\theta}{\rho}\beta^2 + \dfrac{2\epsilon^2}{1-\nu}\,\dfrac{(1 + p/\rho^2)}{(1 + p)^2}}{\dfrac{\theta}{\rho}\beta^2(1 + p\rho^2) + \dfrac{2\epsilon^2}{1-\nu}}$$

is smaller than unity if

$$\theta\rho\beta^2 > -\frac{2\epsilon^2}{1-\nu}\,\frac{(2 + p - 1/\rho^2)}{(1 + p)^2} \tag{76}$$

This condition is usually satisfied, as all quantities are nonzero and positive while $1/\rho^2 < 1.5$; that is, the right-hand side of the inequality is negative in many practical cases.

However, it should be noticed that for shells of finite length, this conclusion is too sweeping. In the foregoing analysis, the wavelength

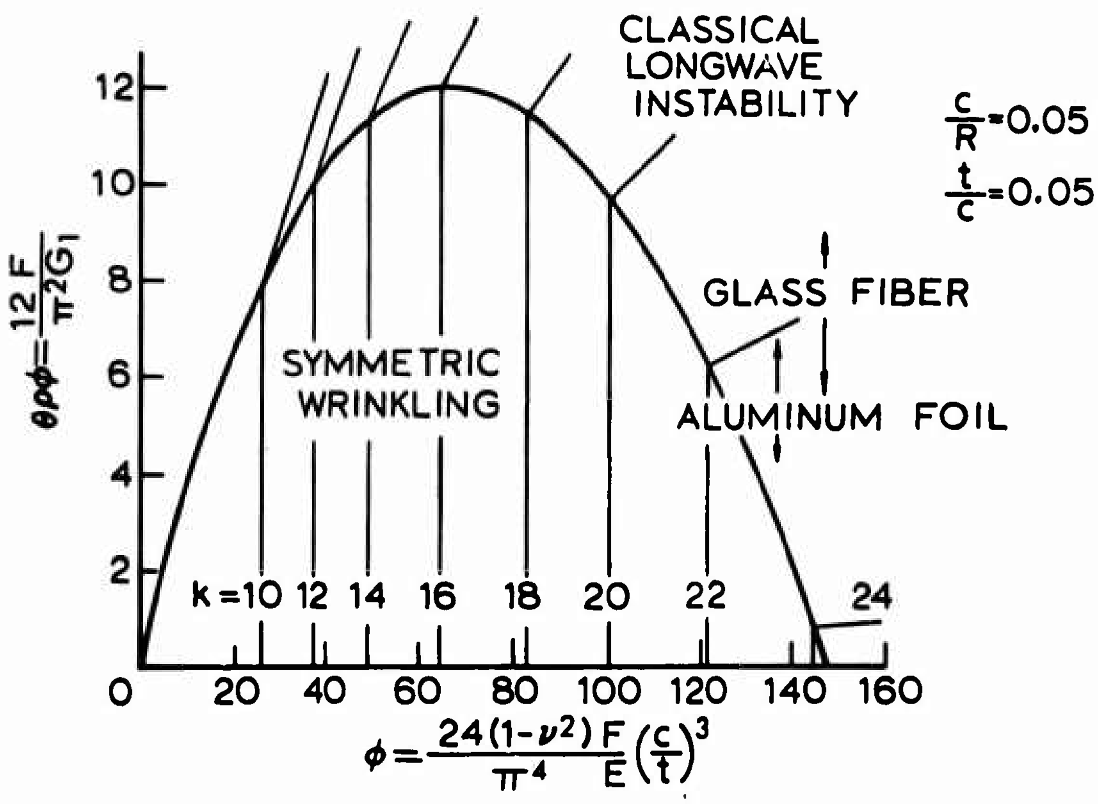


Figure 6. Stability Boundary for Longwave and Symmetric Wrinkling Instability of a Honeycomb Sandwich Cylinder $(c/R = 0.05 , t/c = 0.05)$ .

## TABLE III

## NONAXISYMMETRIC INSTABILITY MODES

| | Longwave Instability Mode ("classical" mode) | Antisymmetric Wrinkling Mode | Symmetric Wrinkling Mode |
|---|---|---|---|
| **Critical wavelength parameter $\beta$ :** | $\frac{\epsilon\zeta^{1/2}}{\sqrt{1+p}}\left[\epsilon(1+\epsilon)(1+p)\sqrt{3\zeta}-f\theta\rho\zeta\right]^{-1/2}$ (Eq. (62)) | $\frac{\left[(1+\epsilon)\sqrt{3\varphi}-f\theta\rho\varphi\right]^{1/2}}{\sqrt{1+p}}$ (Eq. (64)) | $\frac{\varphi^{1/4}}{\sqrt{1+p}}$ (Eq. (57)) |
| **Critical load parameter k :** | $2(1+\epsilon)\sqrt{3\zeta/\epsilon^2}-\frac{f}{1+p}\theta\rho\zeta/\epsilon^2$ (Eq. (63)) | $(1+p)\left[2(1+\epsilon)\sqrt{3\varphi}-\frac{1+p}{1+p\rho^2}\theta\rho\varphi\right]$ (Eq. (65)) | $2(1+p)\sqrt{\varphi}$ (Eq. (58)) |
| **Assumption** | $\beta \ll 1$ | $\beta > 1$ | $\zeta \ll \varphi$ |

parameter $\beta$ is treated as a continuous variable. When the cylinder is assumed to accommodate only an integer number m of halfwaves, then $\beta = (c/L)m$ just takes discrete values. Either one of the two discrete values enclosing the theoretical critical value of a continuous variable $\beta$ may correspond to the lowest value of the load parameter k . This value of the load parameter, of course, will be higher than the minimum value reached in case of a continuous $\beta$ . The magnitude of these deviations from the critical load parameters as given by Equations (46) and (63) can be ascertained only by numerical evaluation of a particular case. However, it can be observed that this effect is small as $\partial k/\partial\beta^2$ is near zero in the vicinity of the critical wavelength. Only in cases where the last term on the right-hand side of Equations (46) and (63) is small with respect to the first term $(\theta\rho\zeta/\epsilon^2 \ll 2(1+\epsilon)\ \ 3\zeta/\epsilon^2)$ may the nonaxisymmetric buckling mode yield a slightly different and possibly lower buckling load. However, in view of other simplifications made, the axisymmetric buckling load can be used to predict the classical buckling mode in most practical cases.

## WEAK CORE SHELLS

The approximate closed-form expressions for the critical loads and wavelengths of the two antisymmetric buckling modes are derived on the basis of assumptions with respect to the order of magnitude of the wavelength parameter $\beta$ . For the classical mode, $\beta$ is assumed to be much smaller than unity. In order to assess the effect of the core stiffness on the critical wavelength, the axisymmetric case is considered (Table II). It is seen that for the classical mode,

$$\frac{\delta\beta}{\delta(\theta\rho)} = -\frac{c}{\lambda^2}\frac{\delta\lambda}{\delta(\theta\rho)} = \frac{\beta^3}{2\epsilon^2} > 0 \qquad (77)$$

The quantity $\theta\rho$ is proportional to $1/G_1$ . For decreasing core stiffness, $d(\theta\rho) > 0$ and, hence, $d\lambda < 0$ . If $\theta\rho$ approaches the value $\epsilon(1+\epsilon)\sqrt{3/\zeta}$ , then the wavelength parameter tends to infinity (the wavelength tends to zero) and the approximation is no longer valid.

This degenerate case is the classical "shear-failure" phenomenon, with a critical load parameter

$$\left(k_{cr}\right)_{cl} = 3(1 + \epsilon)^2/\theta\rho \tag{78}$$

For the shortwave antisymmetric wrinkling mode, the wavelength parameter $\beta$ is assumed to be of the order of magnitude of unity or larger. To evaluate the effect of core shear stiffness in this case, it must be noticed that the quantity $\theta\rho\varphi$ is proportional to $F/G_1$ and that this ratio does not vary appreciably for a given type of core material. Then,

$$\frac{\partial\beta}{\partial\varphi} = -\frac{c}{\lambda^2}\frac{\partial\lambda}{\partial\varphi} = \frac{\sqrt{3}}{4\sqrt{\varphi}}\frac{1 + \epsilon}{\beta} > 0 \tag{79}$$

But as $d\varphi < 0$ for a decreasing core stiffness, $d\lambda > 0$. If with decreasing core stiffness $(1 + \epsilon)\sqrt{3\varphi} - \theta\rho\varphi \rightarrow 0$, then $(\beta_{cr})_2 \rightarrow 0$ (the wavelength increases beyond bounds) and the assumption $\beta = O(1)$ is violated.

Clearly, the wavelengths of the two antisymmetric buckling modes approach each other as the core stiffness decreases, and they eventually coalesce (Figure 5). Which of the two critical load formulas (Equations (46) and (49)) best approximates the minimum value depends on the magnitude of $\beta$ for which the coalescence occurs. The "shear-failure" formula (Equation (78)) constitutes a lower bound in any case.

The criteria developed herein are utilized to predict the buckling loads of the sandwich cylinders constructed and tested as reported in References 10, 18, and 19. In the first and last, axial compression tests were carried out; in the second, the cylinders were tested in bending. With regard to the bending tests, the cylinders reflect thickness-to-radius ratios and radius-to-length ratios sufficiently large to permit the use of the axial compression buckling criteria.

The results of the comparison of theoretical prediction with experiment is presented in Table IV. The theoretical buckling stresses are obtained, as appropriate, from either Equations (43), (46), and (49) or their reduced-stiffness counterparts developed in Appendix IV as Equations (125), (127), and (129). In view of the fact that all of the cylinders tested demonstrate practical construction and that the bending tests were somewhat affected by end conditions, it would appear that good agreement has been realized. The implication, based on the limited test data, is that buckling stresses for sandwich cylinders with geometric and material parameters in the practical range of interest are given in good approximation by linear theory with no apparent need for the excessive "knock-down" factors appearing in current design criteria. The ratio of experimental and theoretically predicted critical stress, $\eta$ , based on the data of References 10, 18, and 19 and the present theory (see Table IV), is shown in Figure 7. For comparison, design recommendations applicable to general instability of initially imperfect, linear elastic shells (see References 20 and 21) are included in Figure 7.

TABLE IV

COMPARISON OF THEORETICAL AND TEST RESULTS FOR BUCKLING OF SANDWICH CYLINDERS

| Specimen | Thickness Ratio (c/R) | Theoretical Critical Stress (ksi) (axisymm. buckling) General Instability (Eq.(46) or (127)) I | Antisymm. Wrinkling (Eq.(49) or (129)) II | Symm. Wrinkling (Eq.(43) or (125)) III | Failure Stress (ksi) | $\eta = \frac{\sigma_{test}}{(\sigma_{cr})_{theory}}$ | Failure Mode |
|---|---|---|---|---|---|---|---|
| [18] | | | | | | | |
| 1 | $0.57 \times 10^{-2}$ | 64.0 | 76.4 | 77.4 | 51.5 1) | 0.81 | I |
| 2 | $0.80 \times 10^{-2}$ | 69.2 | 72.9 | 72.9 | 60.0 | 0.87 | I |
| 3 | $1.04 \times 10^{-2}$ | 66.3 | 68.5 | 68.5 | 58.0 | 0.88 | I |
| [10] | | | | | | | |
| 1 | $0.64 \times 10^{-2}$ | 66.4 | 75.6 | 76.9 | 59.6 2) | 0.89 | edge wrinkle |
| 2 | $1.05 \times 10^{-2}$ | 73.5 | 76.4 | 76.4 | 67.4 | 0.91 | II |
| 3 | $1.05 \times 10^{-2}$ | 73.5 | 76.4 | 76.4 | 60.9 | 0.82 | II |
| 4 | $2.22 \times 10^{-2}$ | 77.4 | 76.4 | 75.6 | 70.3 | 0.92 | III |
| 5 | $2.22 \times 10^{-2}$ | 77.4 | 76.4 | 75.6 | 74.8 | 0.97 | III |
| [19] | | | | | | | |
| 1 | $1.09 \times 10^{-2}$ | 68.4 3) | 74.6 | 71.6 | 62.5 | 0.91 | I |

TABLE IV (Continued)

| Specimen | Thickness Ratio (c/R) | Theoretical Critical Stress (ksi) (axisymm. buckling) General Instability (Eq.(46) or (127)) I | Antisymm. Wrinkling (Eq.(49) or (129)) II | Symm. Wrinkling (Eq.(43) or (125)) III | Failure Stress (ksi) | $\eta = \frac{\sigma_{test}}{(\sigma_{cr})_{theory}}$ | Failure Mode |
|---|---|---|---|---|---|---|---|
| [19] | | | | | | | |
| 2 | $0.73 \times 10^{-2}$ | 61.4 | 74.8 | 72.9 | 56.7 | 0.92 | I |
| 3 | $0.73 \times 10^{-2}$ | 61.4 | 74.8 | 73.2 | 48.7 | 0.79 | I |
| 4 | $0.73 \times 10^{-2}$ | 61.4 | 74.8 | 73.3 | 48.3 | 0.79 | I |
| 5 | $0.46 \times 10^{-2}$ | 50.0 | 74.8 | 73.6 | 26.0 | 0.52 | I |

NOTES

1) Bending tests – Critical stress determined from back-to-back strain gage measurements and uniaxial stress-strain curve of face material.

2) Axial compression tests – Critical stress determined from failure load assuming core effectivity, in carrying inplane loads, of 25% of theoretical stiffness.

3) Axial compression tests – Critical stress based on stress-strain relationship determined by author[19] using coupons of nominally identical material with same ultimate strength.

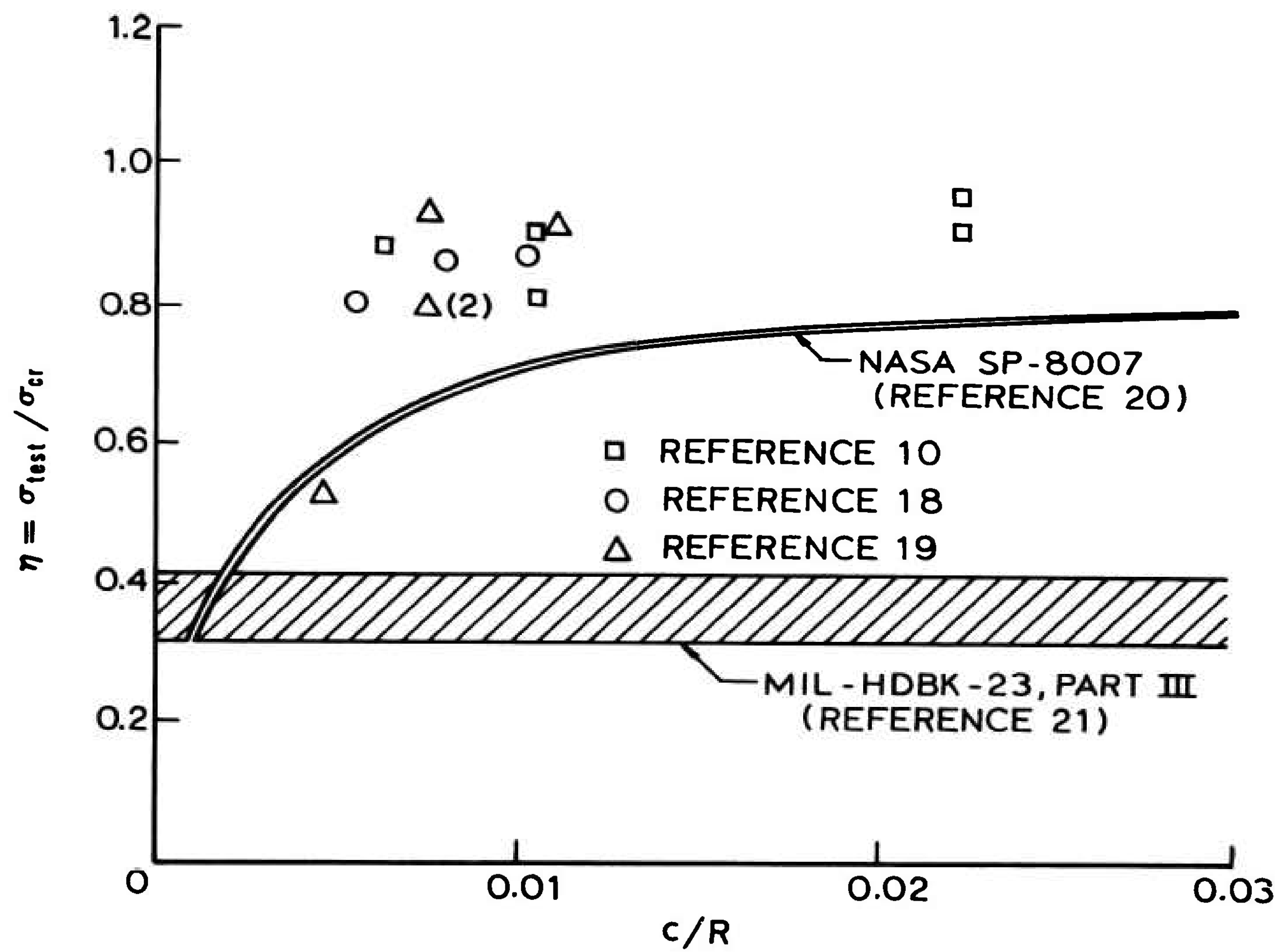


Figure 7. Ratio of Experimental and Theoretical Buckling Stresses for Perfect Shells (Table IV) Compared with Design Recommendations for General Instability of Linear Elastic, Imperfect Shells (References 20 and 21).

## CONCLUSIONS

This report presents the governing equations and boundary conditions for moderately large deflections of a sandwich shell of arbitrary shape. Equilibrium equations and boundary conditions for both the thin-face layers and the thick-core layers are consistently derived from the condition for a stationary value of the total potential. For the core, the face parallel stresses are neglected; but the transverse strain is included, and, unlike previous theories employing the same degrees of freedom, the relevant strain displacement relations for the core are linearized. The stiffness parameters of the core are assumed to vary linearly through the thickness, as is the case in a curved-core layer of cell structure. In Reference 5, not only linear terms in the normal coordinate are retained but also higher order terms, while the stiffness parameters are assumed to be constant. The present approach is believed to provide a more consistent treatment of the core. These assumptions allow the three equilibrium equations of the core equation to be integrated with respect to the normal coordinate; all loads transferred by the core can be expressed in terms of the components of the face displacements and two core-shear angles, while two compatibility equations are obtained as auxiliary conditions.

As a final set, three equilibrium equations and four boundary conditions for the edge of each of the face layers are presented in terms of the face-displacement components. They are of the type usually obtained on the basis of the Kirchhoff-Love assumptions for a thin shell.

In addition, two compatibility conditions and one boundary condition for the core are given also in terms of the face displacements. The eight governing equations and the nine boundary conditions, then, are functions of two surface coordinates only.

Linearization leads to a set of equations with a symmetric operator matrix as is appropriate for a linear elastic system. The order of a single governing equation, as derived from the operator matrix of the full set, is 18, with 9 boundary conditions which can be specified at an edge in terms of either generalized force or displacement.

The specification of a consistent set of boundary conditions is necessary not only for the derivation of an exact or approximate solution to a problem but also for a meaningful comparison of theoretical predictions and the results of appropriate tests. Although the nine sets of boundary conditions occur in a discouragingly large number of possible combinations, practical designs will limit the number of combinations and should, in addition, stimulate the analysis of important boundary situations other than the very limited set that is usually considered.

The unified theory is used to estimate the range of applicability of classical theories or simplified versions of the present theory. In an example, the critical loads for an axially compressed circular cylinder are determined.

In general, three instability modes are possible:

1. An instability pattern with a wavelength of several times the shell thickness and negligible transverse strain. This mode corresponds to the results obtained from the classical sandwich theory, ignoring core normal strain.
2. A face-wrinkling pattern of very short wavelength (of the order of the shell thickness) and characterized by essentially similar face displacements (antisymmetric wrinkling).
3. A face-wrinkling mode of comparable wavelength but face displacements that are essentially symmetric with respect to the middle surface of the shell (symmetric wrinkling).

By means of a parameter study, approximate expressions for the critical loads and wavelengths of all three instability modes are derived. The limits of applicability are indicated. The study also reveals that the symmetric wrinkling mode will, for most practical sandwiches, lead to the lowest wrinkling load. In Reference 3, the opposite conclusion is reached; however, it appears that the results of numerical analysis are not consistent with the basic theory.

The parameter analysis further discloses that, with decreasing core stiffness, the wavelengths of the two antisymmetric modes (wrinkling and overall instability) approach each other and finally coincide. The coalescence of critical wavelengths can occur near the overall instability wavelengths as well as near the wrinkling wavelength. Only in the first case is the classical "shear failure" formula a good approximation.

The critical loads derived from a nonaxisymmetric deformation pattern may be lower in case of the longwave deformation, but it is shown that the difference will be small.

A comparison with limited experiments suggests that critical loads derived from a linear analysis of axially compressed sandwich cylinders should be used to predict the buckling loads of such cylinders. The present analysis shows also that a consideration of the discrete core structure may be necessary to account for existing discrepancies between linear theory and experiment. However, unlike the case of homogeneous, isotropic shells, the discrepancies that do exist appear to be more a function of core deformation prior to buckling and inelastic behavior at buckling rather than the result of overall imperfections in the sandwich shell wall.

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## APPENDIX I

## VARIATIONAL DERIVATION OF THE GOVERNING EQUATIONS FOR FACE SHEETS

The face sheets are treated as thin, shallow shells of linearly elastic isotropic material. The governing equations are derived from the condition for a stationary value of the total potential. The total potential is defined as the sum of the strain energy function and a potential function of the applied loads, as appropriate.

### THE STRAIN ENERGY FUNCTION

The appropriate strain energy density expression for linearly elastic thin shells, consistent with the Kirchhoff-Love assumptions, is:[7]

$$W_{f_i} = \frac{C_i}{2}\left[\epsilon_1^2 + \epsilon_2^2 + 2\nu\epsilon_1\epsilon_2 + \frac{1-\nu}{2}\gamma_{12}^2\right]_i + \frac{D_i}{2}\left[\kappa_1^2 + \kappa_2^2 + 2\nu\kappa_1\kappa_2 + 2(1-\nu)\kappa_{12}^2\right]_i \qquad (i = 1,2) \tag{80}$$

For the case of small strains and moderate out-of-plane rotations, the inplane rotation about the normal, determined by the quantity

$$\Omega = \frac{1}{2}\left[\frac{1}{\alpha_1} v,_{\xi_1} - \frac{1}{\alpha_2} u,_{\xi_2} - \frac{u}{\alpha_1\alpha_2}\alpha_{1,\xi_2} + \frac{v}{\alpha_1\alpha_2}\alpha_{2,\xi_1}\right]$$

is neglected with respect to the rotations of the normal to the middle surface given by

$$\omega_1 = \frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1} \quad \text{and} \quad \omega_2 = \frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2} \tag{81}$$

The middle surface strains are then approximated by

$$\epsilon_1 = \frac{1}{\alpha_1} u,_{\xi_1} + \frac{v}{\alpha_1\alpha_2}\alpha_{1,\xi_2} + \frac{w}{\rho_1} + \frac{1}{2}\left[\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right]^2$$

$$\epsilon_2 = \frac{1}{\alpha_2} v,_{\xi_2} + \frac{u}{\alpha_1\alpha_2}\alpha_{2,\xi_1} + \frac{w}{\rho_2} + \frac{1}{2}\left[\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right]^2$$

$$2\epsilon_{12} = \gamma_{12} = \frac{1}{\alpha_1} v,_{\xi_1} + \frac{1}{\alpha_2} u,_{\xi_2} - \frac{u}{\alpha_1\alpha_2}\alpha_{1,\xi_2} - \frac{v}{\alpha_1\alpha_2}\alpha_{2,\xi_1}$$

$$+ \left[\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right]\left[\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right] \qquad (82)$$

The geometry is shown in Figure 1; the coordinate directions 1 and 2 are directions of the principal curvature; the principal curvatures are $1/\rho_1$ and $1/\rho_2$ , respectively. The changes of curvature and twist of the deformed middle surface are

$$\kappa_1 = \frac{1}{\alpha_1}\left[\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right],_{\xi_1} + \frac{1}{\alpha_1\alpha_2}\left[\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right]\alpha_{1,\xi_2}$$

$$\kappa_2 = \frac{1}{\alpha_2}\left[\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right],_{\xi_2} + \frac{1}{\alpha_1\alpha_2}\left[\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right]\alpha_{2,\xi_1}$$

$$2\kappa_{12} = \frac{1}{\alpha_1}\left[\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right],_{\xi_1} - \frac{1}{\alpha_1\alpha_2}\left[\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right]\alpha_{1,\xi_2}$$

$$+ \frac{1}{\alpha_2}\left[\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right],_{\xi_2} - \frac{1}{\alpha_1\alpha_2}\left[\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right]\alpha_{2,\xi_1} \qquad (83)$$

The total strain energy of a face sheet is then

$$U_{f_i} = \iint_D \left[W_f\alpha_1\alpha_2\right]_i d\xi_1 d\xi_2 \qquad (i = 1 \text{ or } 2) \qquad (84)$$

## POTENTIAL OF APPLIED LOADS

The boundary loads on a face sheet are indicated in Figure 2. Uppercase letters denote loads per unit length; lowercase letters denote loads per unit area.

For the variational problem, as the equivalent of the principle of virtual displacements for an equilibrium system, a potential of applied loads V is appropriately defined as:

$$V_i = - \oint_{\partial S} \left\{ \left[ N_n d_n + N_s d_s + N_z w + M_n \omega_n - M_{ns} \omega_s \right] \alpha_s \right\}_i d\xi_s$$
$$\pm \iint_S \left\{ \left[ q_1 \left( u \pm \frac{t}{2} \omega_1 \right) + q_2 \left( v \pm \frac{t}{2} \omega_2 \right) + \left( q_3 - p \right) w \right] \right.$$
$$\left. \cdot \alpha_1 \alpha_2 \right\}_i d\xi_1 \, d\xi_2 \qquad (85)$$

in which

$$\omega_s = \frac{1}{\alpha_s} w,_{\xi_s} - \frac{d_n}{\tau} - \frac{d_s}{\rho_s}$$

$$\omega_n = \frac{1}{\alpha_n} w,_{\xi_n} - \frac{d_s}{\tau} - \frac{d_n}{\rho_n}$$

The total potential function is then defined by

$$T_i = U_i + V_i \qquad (86)$$

It should be noted that the neglect of the inplane rotations and the product terms in the potential expression involving the out-of-plane rotations leads to Euler equations of a first variation that are incorrect in terms involving rotation components. Adjustments will be made later to obtain the simpler form of equilibrium equations for coordinate directions on the deformed surface rather than for the original coordinate directions.

Variation of the Total Potential

For an equilibrium system, the first variation of the total potential vanishes for any set of variations of the argument functions compatible with imposed boundary conditions. Carrying out the first variation and integrating by parts integrals containing derivatives of the argument functions $u_i$ , $v_i$ , $w_i$ leads to

$$\delta T = \iint_S \delta u \Bigg[ - \left\{ \alpha_2 \, C(\epsilon_1 + \nu\epsilon_2) \right\}_{,\xi_1} + C(\epsilon_2 + \nu\epsilon_1)\, \alpha_{2,\xi_1}$$

$$- \left\{ \alpha_1 \, C \frac{1-\nu}{2} \gamma_{12} \right\}_{,\xi_2} - C \frac{1-\nu}{2} \gamma_{12}\, \alpha_{1,\xi_2} \pm q_1 \alpha_1 \alpha_2$$

$$- \alpha_1\alpha_2 \frac{C}{\rho_1} (\epsilon_1 + \nu\epsilon_2) \left( \frac{1}{\alpha_1} w_{,\xi_1} - \frac{u}{\rho_1} \right)$$

$$- \alpha_1\alpha_2 \frac{C}{\rho_1} \frac{1-\nu}{2} \gamma_{12} \left( \frac{1}{\alpha_2} w_{,\xi_2} - \frac{v}{\rho_2} \right)$$

$$+ \frac{1}{\rho_1} \left\{ \alpha_2 \, D(\kappa_1 + \nu\kappa_2) \right\}_{,\xi_1} - \frac{1}{\rho_1} D(\kappa_2 + \nu\kappa_1)\, \alpha_{2,\xi_1}$$

$$+ \frac{1}{\rho_1} \left\{ \alpha_1 \, D(1-\nu)\, \kappa_{12} \right\}_{,\xi_2} + \frac{1}{\rho_1} D(1-\nu)\, \kappa_{12} \alpha_{1,\xi_2} \Bigg] d\xi_1 d\xi_2$$

$$+ \iint_S \delta v \Bigg[ - \left\{ \alpha_1 \, C(\epsilon_2 + \nu\epsilon_1) \right\}_{,\xi_2} + C(\epsilon_1 + \nu\epsilon_2)\, \alpha_{1,\xi_2}$$

$$- \left\{ \alpha_2 \, C \frac{1-\nu}{2} \gamma_{12} \right\}_{,\xi_1} - C \frac{1-\nu}{2} \gamma_{12}\, \alpha_{2,\xi_1} \pm q_2 \alpha_1 \alpha_2 \qquad \text{(Cont'd)}$$

$$
\begin{aligned}
& - \alpha_1\alpha_2 \frac{C}{\rho_2} (\epsilon_2 + \nu\epsilon_1)\left(\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right) \\
& - \alpha_1\alpha_2 \frac{C}{\rho_2} \frac{1-\nu}{2} \gamma_{12}\left(\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right) \\
& + \frac{1}{\rho_2}\left\{\alpha_1 D(\kappa_2 + \nu\kappa_1)\right\},_{\xi_2} - \frac{1}{\rho_2} D(\kappa_1 + \nu\kappa_2)\, \alpha_{1,\xi_2} \\
& + \frac{1}{\rho_2}\left\{\alpha_2 D(1-\nu)\, \kappa_{12}\right\},_{\xi_1} + \frac{1}{\rho_2} D(1-\nu)\, \kappa_{12}\alpha_{2,\xi_1}\Bigg] d\xi_1 d\xi_2 \\
& + \iint_S \delta w \Bigg[\Bigg\{\frac{1}{\alpha_1}\Bigg[\left\{\alpha_2 D(\kappa_1 + \nu\kappa_2)\right\},_{\xi_1} - D(\kappa_2 + \nu\kappa_1)\, \alpha_{2,\xi_1} \\
& + \left\{\alpha_1 D(1-\nu)\, \kappa_{12}\right\},_{\xi_2} + D(1-\nu)\, \kappa_{12}\alpha_{1,\xi_2}\Bigg]\Bigg\},_{\xi_1} \\
& + \Bigg\{\frac{1}{\alpha_2}\Bigg[\left\{\alpha_1 D(\kappa_2 + \nu\kappa_1)\right\},_{\xi_2} - D(\kappa_1 + \nu\kappa_2)\, \alpha_{1,\xi_2} \\
& + \left\{\alpha_2 D(1-\nu)\, \kappa_{12}\right\},_{\xi_1} + D(1-\nu)\, \kappa_{12}\alpha_{2,\xi_1}\Bigg]\Bigg\},_{\xi_2} \\
& + \alpha_1\alpha_2 \frac{C(\epsilon_1 + \nu\epsilon_2)}{\rho_1} + \alpha_1\alpha_2 \frac{C(\epsilon_2 + \nu\epsilon_1)}{\rho_2} \\
& - \left\{\alpha_2 C(\epsilon_1 + \nu\epsilon_2)\left(\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right)\right\},_{\xi_1} \\
& - \left\{\alpha_1 C(\epsilon_2 + \nu\epsilon_1)\left(\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right)\right\},_{\xi_2}
\end{aligned}
$$

(Cont'd)

$$- \left\{ \alpha_1 \, C \, \frac{1-\nu}{2} \, \gamma_{12} \left( \frac{1}{\alpha_1} \, w,_{\xi_1} - \frac{u}{\rho_1} \right) \right\},_{\xi_2}$$

$$- \left\{ \alpha_2 \, C \, \frac{1-\nu}{2} \, \gamma_{12} \left( \frac{1}{\alpha_2} \, w,_{\xi_2} - \frac{v}{\rho_2} \right) \right\},_{\xi_1}$$

$$- \frac{t}{2} \left\{ (\alpha_1 q_1),_{\xi_1} + (\alpha_1 q_2),_{\xi_2} \right\} \pm \left\{ - p + q_3 \right\} \alpha_1 \alpha_2 \Bigg] \, d\xi_1 d\xi_2$$

$$+ \oint_{\partial S} \delta u \left[ C(\epsilon_1 + \nu \epsilon_2) \, \alpha_2 d\xi_2 + C \, \frac{1-\nu}{2} \, \gamma_{12} \alpha_1 d\xi_1 \right.$$

$$\left. - \frac{D}{\rho_1} \, (1-\nu) \, \kappa_{12} \alpha_1 d\xi_1 - \frac{D}{\rho_1} \, (\kappa_1 + \nu \kappa_2) \, \alpha_2 d\xi_2 \right]$$

$$+ \oint_{\partial S} \delta v \left[ C(\epsilon_2 + \nu \epsilon_1) \, \alpha_1 d\xi_1 + C \, \frac{1-\nu}{2} \, \gamma_{12} \alpha_2 d\xi_2 \right.$$

$$\left. - \frac{D}{\rho_2} \, (\kappa_2 + \nu \kappa_1) \, \alpha_1 d\xi_1 - \frac{D}{\rho_2} \, (1-\nu) \, \kappa_{12} \alpha_2 d\xi_2 \right]$$

$$+ \oint_{\partial S} \delta d_n \left[ \left( N_n - \frac{M_n}{\rho_n} + \frac{M_{ns}}{\tau} \right) \alpha_s d\xi_s \right]$$

$$+ \oint_{\partial S} \delta d_s \left[ \left( N_s - \frac{M_n}{\tau} + \frac{M_{ns}}{\rho_s} \right) \alpha_s d\xi_s \right]$$

$$+ \oint_{\partial S} \delta w \left[ \left\{ C(\epsilon_1 + \nu \epsilon_2) \left( \frac{1}{\alpha_1} \, \delta w,_{\xi_1} - \frac{u}{\rho_1} \right) \right. \right.$$

$$+ C \, \frac{(1-\nu)}{2} \, \gamma_{12} \left( \frac{1}{\alpha_2} \, w,_{\xi_2} - \frac{v}{\rho_2} \right) \Bigg\} \, \alpha_2 d\xi_2 \qquad \text{(Cont'd)}$$

$$
\begin{aligned}
&+ \Big\{ C(\epsilon_2 + \nu\epsilon_1)\left(\frac{1}{\alpha_2}\, w,_{\xi_2} - \frac{v}{\rho_2}\right) + C\,\frac{1-\nu}{2}\,\gamma_{12}\left(\frac{1}{\alpha_1}\, w,_{\xi_1} - \frac{u}{\rho_1}\right)\Big\}\, \alpha_1 d\xi_1 \\
&- \frac{1}{\alpha_1\alpha_2}\Bigg[\Big\{\alpha_2\, D(\kappa_1 + \nu\kappa_2)\Big\},_{\xi_1} - D(\kappa_2 + \nu\kappa_1)\, \alpha_{2,\xi_1} \\
&+ \Big\{\alpha_1\, D(1-\nu)\, \kappa_{12}\Big\},_{\xi_2} + D(1-\nu)\, \kappa_{12}\alpha_{1,\xi_2}\Bigg]\alpha_2 d\xi_2 \\
&- \frac{1}{\alpha_1\alpha_2}\Bigg[\Big\{\alpha_1\, D(\kappa_2 + \nu\kappa_1)\Big\},_{\xi_2} - D(\kappa_1 + \nu\kappa_2)\, \alpha_{1,\xi_2} \\
&+ \Big\{\alpha_2\, D(1-\nu)\, \kappa_{12}\Big\},_{\xi_1} + D(1-\nu)\, \kappa_{12}\alpha_{2,\xi_1}\Bigg]\alpha_1 d\xi_1 \\
&+ \frac{t}{2}\, q_1\alpha_2 d\xi_2 + \frac{t}{2}\, q_2\alpha_1 d\xi_1 - \Big\{N_z - \frac{1}{\alpha_s}\, M_{ns,\xi_s}\Big\}\, \alpha_s d\xi_s\Bigg] \\
&+ \oint_{\partial S}\Bigg[\frac{1}{\alpha_1}\, \delta w,_{\xi_1}\Big\{D(\kappa_1 + \nu\kappa_2)\, \alpha_2 d\xi_2 + D(1-\nu)\, \kappa_{12}\alpha_1 d\xi_1\Big\} \\
&+ \frac{1}{\alpha_2}\, \delta w,_{\xi_2}\Big\{D(\kappa_2 + \nu\kappa_1)\, \alpha_1 d\xi_1 + D(1-\nu)\, \kappa_{12}\alpha_2 d\xi_2\Big\} \\
&+ \frac{1}{\alpha_n}\, \delta w,_{\xi_n}\, M_n\alpha_s d\xi_s\Bigg] \equiv 0 \qquad (87)
\end{aligned}
$$

The potential term involving the twisting moment $M_n$ has been integrated by parts once with respect to the circumferential coordinates S . The contour is assumed to be smooth; if the contour contains singularities (corners), the integration by parts also gives rise to terms that are the product of a variation in the lateral displacement w and a concentrated generalized force ("corner reaction"). It is well known that this identity implies that each of the integrals must vanish identically.

As the variations $\delta u$ , $\delta v$ , and $\delta w$ are completely arbitrary within the domain of integration $S$ , the condition $\delta T \equiv 0$ leads to the following set of Euler equations and boundary conditions. Appropriate quantities have to be assigned a subscript 1 or 2 . Where a double sign appears, the top sign is to be used in connection with the subscript 1 ; the lower, with the subscript 2 . In $S$ , then,

$$\Bigg[\Bigg[\left\{\alpha_2 C(\epsilon_1 + \nu\epsilon_2)\right\}_{,\xi_1} - C(\epsilon_2 + \nu\epsilon_1)\,\alpha_{2,\xi_1} + \left\{\alpha_1 C\frac{1-\nu}{2}\gamma_{12}\right\}_{,\xi_2}$$

$$+ C\frac{1-\nu}{2}\gamma_{12}\alpha_{1,\xi_2} - \frac{1}{\rho_1}\Bigg[\left\{\alpha_2 D(\kappa_1 + \nu\kappa_2)\right\}_{,\xi_1}$$

$$- D(\kappa_2 + \nu\kappa_1)\,\alpha_{2,\xi_1} + \left\{\alpha_1 D(1-\nu)\,\kappa_{12}\right\}_{,\xi_2}$$

$$+ D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2}\Bigg] \mp \alpha_1\alpha_2 q_1\Bigg]_i = 0 \qquad (88a)$$

$$\Bigg[\Bigg[\left\{\alpha_1 C(\epsilon_2 + \nu\epsilon_1)\right\}_{,\xi_2} - C(\epsilon_1 + \nu\epsilon_2)\,\alpha_{1,\xi_2} + \left\{\alpha_2 C\frac{1-\nu}{2}\gamma_{12}\right\}_{,\xi_1}$$

$$+ C\frac{1-\nu}{2}\gamma_{12}\alpha_{2,\xi_1} \mp \alpha_1\alpha_2 q_2 - \frac{1}{\rho_2}\Bigg[\left\{\alpha_1 D(\kappa_2 + \nu\kappa_1)\right\}_{,\xi_2}$$

$$- D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2} + \left\{\alpha_2 D(1-\nu)\,\kappa_{12}\right\}_{,\xi_1}$$

$$+ D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1}\Bigg]\Bigg] = 0 \qquad (88b)$$

$$\left[\left\{\frac{1}{\alpha_1}\left[\left\{\alpha_2 D(\kappa_1 + \nu\kappa_2)\right\}_{,\xi_1} - D(\kappa_2 + \nu\kappa_1)\,\alpha_{2,\xi_1} + \left\{\alpha_1 D(1-\nu)\,\kappa_{12}\right\}_{,\xi_2}\right.\right.\right.$$

$$\left.\left. + D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2}\right]\right\}_{,\xi_1}$$

$$+ \left\{\frac{1}{\alpha_2}\left[\left\{\alpha_1 D(\kappa_2 + \nu\kappa_1)\right\}_{,\xi_2} - D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2}\right.\right.$$

$$\left.\left. + \left\{\alpha_2 D(1-\nu)\,\kappa_{12}\right\}_{,\xi_1} + D(1-\nu)\,\kappa_{12}\alpha_{2,\xi_1}\right]\right\}_{,\xi_2}$$

$$+ \alpha_1\alpha_2\left[C(\epsilon_1 + \nu\epsilon_2)\left(\frac{1}{\rho_1} - \kappa_1\right)\right.$$

$$\left. + C(\epsilon_2 + \nu\epsilon_1)\left(\frac{1}{\rho_2} - \kappa_2\right) - 2C\,\frac{1-\nu}{2}\,\gamma_{12}\kappa_{12}\right]$$

$$\pm \alpha_1\alpha_2\,[q_3 - p]$$

$$\left. - \frac{t}{2}\left[\left(\alpha_2 q_1\right)_{,\xi_1} + \left(\alpha_1 q_2\right)_{,\xi_2}\right]\right]_i = 0 \qquad (88c)$$

**On the boundary** $\delta S$ ,

$$\left[C(\epsilon_1 + \nu\epsilon_2)\,c_{11}c_{21} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{11}^2 + C(\epsilon_2 + \nu\epsilon_1)\,c_{21}c_{11} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{21}^2\right.$$

$$- D(1-\nu)\,\kappa_{12}\,\frac{c_{11}^2}{\rho_1} - D(\kappa_1 + \nu\kappa_2)\,\frac{c_{11}c_{21}}{\rho_1} - D(\kappa_2 + \nu\kappa_1)\,\frac{c_{21}c_{11}}{\rho_2}$$

$$\left. - D(1-\nu)\,\kappa_{12}\,\frac{c_{21}^2}{\rho_2} + N_s - \frac{M_n}{\tau} + \frac{M_{ns}}{\rho_s}\right]_i = 0 \quad \text{or} \quad \delta d_{si} = 0 \qquad (89a)$$

$$\Big[ C(\epsilon_1 + \nu\epsilon_2)\, c_{12}c_{21} + C\,\frac{1-\nu}{2}\,\gamma_{12}c_{12}c_{11} + C(\epsilon_2 + \nu\epsilon_1)\, c_{22}c_{11}$$

$$+ C\,\frac{1-\nu}{2}\,\gamma_{12}c_{22}c_{21} - D(1-\nu)\,\kappa_{12}\,\frac{c_{11}c_{12}^2}{\rho_1} - D(\kappa_1 + \nu\kappa_2)\,\frac{c_{12}c_{21}}{\rho_1}$$

$$- D(\kappa_2 + \nu\kappa_1)\, c_{22}c_{11} - D(1-\nu)\,\kappa_{12}\,\frac{c_{21}c_{22}}{\rho_2}$$

$$+ N_n - \frac{M_n}{\rho_n} + \frac{M_{ns}}{\tau} \Big]_i = 0 \quad \text{or} \quad \delta d_{ni} = 0 \tag{89b}$$

$$\Big[ C(\epsilon_1 + \nu\epsilon_2)\left(\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right) c_{21} + C\,\frac{1-\nu}{2}\,\gamma_{12}\left(\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_1}\right) c_{21}$$

$$+ C(\epsilon_2 + \nu\epsilon_1)\left(\frac{1}{\alpha_2} w,_{\xi_2} - \frac{v}{\rho_2}\right) c_{11} + C\,\frac{1-\nu}{2}\,\gamma_{12}\left(\frac{1}{\alpha_1} w,_{\xi_1} - \frac{u}{\rho_1}\right) c_{11}$$

$$- \frac{1}{\alpha_s}\Big[ d_{11}\Big\{ D(\kappa_1 + \nu\kappa_2)\, c_{21} + D(1-\nu)\,\kappa_{12}c_{11} \Big\}$$

$$+ d_{12}\Big\{ D(\kappa_2 + \nu\kappa_1)\, c_{11} + D(1-\nu)\,\kappa_{12}c_{21} \Big\}\Big],_{\xi_s}$$

$$- \frac{1}{\alpha_1\alpha_2}\Big[\Big\{ \alpha_2\, D(\kappa_1 + \nu\kappa_2) \Big\},_{\xi_1} - D(\kappa_2 + \nu\kappa_1)\,\alpha_{2,\xi_1}$$

$$+ \Big\{ \alpha_1\, D(1-\nu)\,\kappa_{12} \Big\},_{\xi_2} + D(1-\nu)\,\kappa_{12}\alpha_{1,\xi_2} \Big] c_{21}$$

$$- \frac{1}{\alpha_1\alpha_2}\Big[\Big\{ \alpha_1\, D(\kappa_2 + \nu\kappa_1) \Big\},_{\xi_2} - D(\kappa_1 + \nu\kappa_2)\,\alpha_{1,\xi_2} \qquad \text{(Cont'd)}$$

$$+ \left\{ \alpha_2 D(1-\nu) \kappa_{12} \right\}_{,\xi_1} + D(1-\nu) \kappa_{12} \alpha_{2,\xi_1} \Bigg] c_{11}$$

$$+ \frac{t}{2} q_1 c_{21} + \frac{t}{2} q_2 c_{11} - N_z + \frac{1}{\alpha_s} M_{ns,\xi_s} \Bigg]_i = 0$$

$$\text{or} \quad \delta w_i = 0 \tag{89c}$$

$$\Bigg[ \left\{ D(\kappa_1 + \nu\kappa_2) c_{21} + D(1-\nu) \kappa_{12} c_{11} \right\} d_{21} + \Big\{ D(\kappa_2 + \nu\kappa_1) c_{11}$$

$$+ D(1-\nu) \kappa_{12} c_{21} \Big\} d_{22} + M_n \Bigg]_i = 0$$

$$\text{or} \quad \delta w_{i,\xi_n} = 0 \tag{89d}$$

The three Euler equations for the variations $\delta u$ , $\delta v$ , and $\delta w$ (88a,b,c) are recognized as the equilibrium equations for components of forces in the three coordinate directions. The last Euler equation has been rewritten, using the first two.

The four sets of boundary conditions (89a,b,c,d) are obtained from the boundary integrals by use of suitable transformations from the $\xi_1, \xi_2$ coordinate system into the $\xi_s, \xi_n$ system. The transformation symbols $c_{\alpha\beta}$ and $d_{\alpha\beta}$ are defined by

$$c_{11} = \frac{\alpha_1}{\alpha_s} \frac{\partial \xi_1}{\partial \xi_s} , \quad c_{12} = \frac{\alpha_1}{\alpha_n} \frac{\partial \xi_1}{\partial \xi_n} , \quad \text{etc.}$$

$$d_{11} = \frac{\alpha_s}{\alpha_1} \frac{\partial \xi_s}{\partial \xi_1} , \quad d_{12} = \frac{\alpha_s}{\alpha_2} \frac{\partial \xi_s}{\partial \xi_2} , \quad \text{etc.} \tag{90}$$

## APPENDIX II

## THE GOVERNING EQUATIONS FOR A CORE MODEL

The analysis of the core is based on the assumption that the inplane shear stiffness and extensional stiffness of the core are negligible with respect to the transverse shear and extensional stiffness quantities.

Further, if the rotation about the normal to the 1-2 coordinate surface is considered to be much smaller than the rotation about tangents to the same surface (consistent with the assumption made for the face sheets), then the essential strain-displacement relations reduce to

$$\gamma_{13} = u,_{\zeta} - \frac{u}{\alpha_1}\alpha_{1,\zeta} + \frac{1}{\alpha_1} w,_{\xi_1} \tag{91a}$$

$$\gamma_{23} = v,_{\zeta} - \frac{v}{\alpha_2}\alpha_{2,\zeta} + \frac{1}{\alpha_2} w,_{\xi_2} \tag{91b}$$

$$\epsilon_3 = w,_{\zeta} + \underline{\frac{1}{2} u^2,_{\zeta} + \frac{1}{2} v^2,_{\zeta}} \tag{91c}$$

For a core with a cell structure consisting of walls perpendicular to the $\zeta = 0$ coordinate surface, the last two terms in the expression for the transverse strain correspond to rotation of cell walls out of their planes. If the squares of these rotations are not negligible with respect to the linear term $w,_{\zeta}$ , then the effects of the stresses and strains in the cell walls, that correspond to a state of plane stress and are neglected a priori, would certainly have to be retained in the equilibrium equations.

Consistent with the initial neglect, therefore, the strain-displacement relations of the core are linearized; that is, the underlined terms in Equation (91c) are omitted.

For a core of cell structure, the appropriate stiffness quantities are

$$F(\zeta)\alpha_1\alpha_2 \;=\; F(0)\bar{\alpha}_1\bar{\alpha}_2 \;\equiv\; F\bar{\alpha}_1\bar{\alpha}_2$$

$$G_1(\zeta)\alpha_2 \;=\; G_1(0)\bar{\alpha}_2 \;\equiv\; G_1\bar{\alpha}_2$$

$$G_2(\zeta)\alpha_1 \;=\; G_2(0)\bar{\alpha}_1 \;\equiv\; G_2\bar{\alpha}_1 \qquad (92)$$

The strain energy of the core is

$$U_c = \iiint\limits_R \left[\frac{1}{2}F(\zeta)\,\epsilon_3^2 + \frac{1}{2}G_1(\zeta)\,\gamma_{13}^2 + \frac{1}{2}G_2(\zeta)\,\gamma_{23}^3\right]\alpha_1\alpha_2 d\xi_1 d\xi_2 d\zeta \qquad (93)$$

and the potential of the applied loads is

$$V_c = -\iint\limits_S \Big[(q_1 u + q_2 v + q_3 w)\,\alpha_1\alpha_2\Big]_{-c/2}^{c/2} d\xi_1 d\xi_2 - \oint\limits_{\partial S}\int_{-c/2}^{c/2} q^* w\alpha_s d\xi_s d\zeta \qquad (94)$$

The total potential of the core is

$$T_c = U_c + V_c \qquad (95)$$

For a stationary character of the total potential, the first variation must vanish identically; thus,

$$\begin{aligned}
\delta T_c = & \iiint\limits_R \delta u \left[-(\alpha_1 G_1\bar{\alpha}_2\gamma_{13})_{,\zeta} - G_1\bar{\alpha}_2\gamma_{13}\alpha_{1,\zeta}\right] d\xi_1 d\xi_2 d\zeta \\
& + \iiint\limits_R \delta v \left[-(\alpha_2 G_2\bar{\alpha}_1\gamma_{23})_{,\zeta} - G_2\bar{\alpha}_1\gamma_{23}\alpha_{2,\zeta}\right] d\xi_1 d\xi_2 d\zeta \\
& + \iiint\limits_R \delta w \left[-F\bar{\alpha}_1\bar{\alpha}_2\, w_{,\zeta\zeta} - (G_1\bar{\alpha}_2\gamma_{13})_{,\xi_1} - (G_2\alpha_1\gamma_{23})_{,\xi_2}\right] d\xi_1 d\xi_2 d\zeta
\end{aligned}$$

$$+ \iint_S \left[ \delta u \cdot \alpha_1\alpha_2 \left\{ G_1 \frac{\bar{\alpha}_2}{\alpha_2} \gamma_{13} - q_1 \right\} \right]_{\zeta=-c/2}^{c/2} d\xi_1 d\xi_2$$

$$+ \iint_S \left[ \delta v \cdot \alpha_1\alpha_2 \left\{ G_2 \frac{\bar{\alpha}_1}{\alpha_1} \gamma_{23} - q_2 \right\} \right]_{\zeta=-c/2}^{c/2} d\xi_1 d\xi_2$$

$$+ \iint_S \left[ \delta w \cdot \alpha_1\alpha_2 \left\{ F \frac{\bar{\alpha}_1\bar{\alpha}_2}{\alpha_1\alpha_2} w_{,\zeta} - q_3 \right\} \right]_{\zeta=-c/2}^{c/2} d\xi_1 d\xi_2$$

$$+ \oint_{\partial S} \int_{-c/2}^{c/2} \delta w \left[ G_1 \frac{\bar{\alpha}_2}{\alpha_2} \gamma_{13}\alpha_2 d\xi_2 + G_2 \frac{\bar{\alpha}_1}{\alpha_1} \gamma_{23} \cdot \alpha_1 d\xi_1 - q^* \alpha_s d\xi_s \right] d\zeta$$

$$\equiv 0 \qquad (96)$$

As the variations $\delta u$ , $\delta v$ , and $\delta w$ are arbitrary everywhere inside the domain of integration and in the interfaces, the following Euler equations and boundary conditions result:

$$(\alpha_1 G_1 \bar{\alpha}_2 \gamma_{13})_{,\zeta} + G_1 \bar{\alpha}_2 \gamma_{13} \alpha_{1,\zeta} = \frac{G_1 \bar{\alpha}_2}{\alpha_1} (\alpha_1^2 \gamma_{13})_{,\zeta} = 0$$

$$(\text{in } R)$$

$$(\alpha_2 G_2 \bar{\alpha}_1 \gamma_{23})_{,\zeta} + G_2 \bar{\alpha}_1 \gamma_{23} \alpha_{2,\zeta} = \frac{G_2 \bar{\alpha}_1}{\alpha_2} (\alpha_2^2 \gamma_{23})_{,\zeta} = 0$$

$$w_{,\zeta\zeta} + \frac{1}{F\bar{\alpha}_1\bar{\alpha}_2} \left[ (\bar{\alpha}_2 G_1 \gamma_{13})_{,\xi_1} + (\bar{\alpha}_1 G_2 \gamma_{23})_{,\xi_2} \right] = 0 \qquad (97)$$

$$G_1 \frac{\bar{\alpha}_2}{\alpha_2} \gamma_{13} = q_1$$

$$(\text{in } S,\ \zeta = \pm c/2)$$

$$G_2 \frac{\bar{\alpha}_1}{\alpha_1} \gamma_{23} = q_2 \qquad (98)$$

$$F\,\frac{\bar{\alpha}_1\bar{\alpha}_2}{\alpha_1\alpha_2}\,w_{,\zeta} = q_3$$

$$G_1\,\frac{\bar{\alpha}_2}{\alpha_2}\,\gamma_{13}\alpha_2 d\xi_2 + G_2\,\frac{\bar{\alpha}_1}{\alpha_1}\,\gamma_{23}\alpha_1 d\xi_1 = -\,q^*\alpha_s d_s \quad \text{or} \quad \delta w = 0$$

$$(\text{on } \partial S\ ,\ |\zeta| < c/2) \qquad (99)$$

The Euler equations can be integrated. The first two equations yield

$$\gamma_{13}(\zeta)\alpha_1^2(\zeta) = \gamma_{13}(0)\alpha_1^2(0) = \gamma_1\bar{\alpha}_1^2$$

$$\gamma_{23}(\zeta)\alpha_2^2(\zeta) = \gamma_{23}(0)\alpha_2^2(0) = \gamma_2\bar{\alpha}_2^2$$

Further, as $\alpha_i(\zeta) = \alpha_i(0)(1 + \zeta/\rho_i)$ , these results can be written as

$$\gamma_{13}(\zeta) = \gamma_1\,(1 - 2\zeta/\rho_1) + O(\zeta^2/\rho_1^2)$$

$$\gamma_{23}(\zeta) = \gamma_2\,(1 - 2\zeta/\rho_2) + O(\zeta^2/\rho_2^2) \qquad (100)$$

The shear-stress distribution across the thickness of the core model is completely determined by the equilibrium equations.

The last Euler equation can be integrated twice with respect to $\zeta$ . With the use of Equation (100), there results

$$w(\xi_1,\xi_2,\zeta) = \left(\frac{\zeta^3}{3\rho_1} - \frac{\zeta^2}{2}\right)\frac{G_1}{F\bar{\alpha}_1\bar{\alpha}_2}\,(\bar{\alpha}_2\gamma_1)_{,\xi_1} + \left(\frac{\zeta^3}{3\rho_2} - \frac{\zeta^2}{2}\right)\frac{G_2}{F\bar{\alpha}_2\bar{\alpha}_1}\,(\bar{\alpha}_1\gamma_2)_{,\xi_2}$$

$$+\ f(\xi_1,\xi_2)\,\zeta + h(\xi_1,\xi_2)$$

Terms of the order $\zeta^2/\rho_1^2$ have consistently been neglected with respect to unity.

The two functions $f(\xi_1,\xi_2)$ and $h(\xi_1,\xi_2)$ can be determined by making use of the "boundary conditions"

$$\begin{aligned} w(\xi_1, \xi_2, c/2) &= w_1 \\ w(\xi_1, \xi_2, -c/2) &= w_2 \end{aligned} \qquad (101)$$

These relations actually express continuity of the lateral displacement $w$ of the interfaces, as the lateral strain in the faces is neglected. By use of Equation (101), there results

$$\begin{aligned} w(\xi_1,\xi_2,\zeta) = {} & \frac{w_1 + w_2}{2} + \frac{w_1 - w_2}{c}\zeta \\ & + \left[\frac{c^2}{8} - \frac{c^2\zeta}{12\rho_1} - \frac{\zeta^2}{2} + \frac{\zeta^3}{3\rho_1}\right] \frac{G_1}{F\bar{\alpha}_1\bar{\alpha}_2} (\bar{\alpha}_2\gamma_1)_{,\xi_1} \\ & + \left[\frac{c^2}{8} - \frac{c^2\zeta}{12\rho_2} - \frac{\zeta^2}{2} + \frac{\zeta^3}{3\rho_2}\right] \frac{G_2}{F\bar{\alpha}_2\bar{\alpha}_1} (\bar{\alpha}_1\gamma_2)_{,\xi_2} \end{aligned} \qquad (102)$$

and, consequently,

$$\begin{aligned} \epsilon_3 = w_{,\zeta} = {} & \frac{w_1 - w_2}{c} - \left(\zeta - \frac{\zeta^2}{\rho_1} + \frac{c^2}{12\rho_1}\right) \frac{G_1}{F\bar{\alpha}_1\bar{\alpha}_2} (\bar{\alpha}_2\gamma_1)_{,\xi_1} \\ & - \left(\zeta - \frac{\zeta^2}{\rho_2} + \frac{c^2}{12\rho_2}\right) \frac{G_2}{F\bar{\alpha}_2\bar{\alpha}_1} (\bar{\alpha}_1\gamma_2)_{,\xi_2} \end{aligned} \qquad (103)$$

From the strain displacement relations (Equations (91a) and (91b)), it is seen that

$$\left(\frac{u}{\alpha_1}\right)_{,\zeta} = \frac{1}{\alpha_1}\gamma_{13} - \frac{1}{\alpha_1^2} w_{,\xi_1}$$

$$\left(\frac{v}{\alpha_2}\right)_{,\zeta} = \frac{1}{\alpha_2}\gamma_{23} - \frac{1}{\alpha_2^2} w_{,\xi_2}$$

Integration of these relations with respect to $\zeta$ , making use of previous results and the continuity conditions,

$$u(\xi_1 , \xi_2 , \tfrac{c}{2}) = u_1 + \frac{t_1}{2}\left(\frac{1}{\alpha_1} w_{1,\xi_1} - \frac{u_1}{\rho_1}\right)$$

$$u(\xi_1,\xi_2, -\tfrac{c}{2}) = u_2 - \frac{t_2}{2}\left(\frac{1}{\alpha_1} w_{2,\xi_1} - \frac{u_2}{\rho_1}\right)$$

$$v(\xi_1 , \xi_2 , \tfrac{c}{2}) = v_1 + \frac{t_1}{2}\left(\frac{1}{\alpha_2} w_{1,\xi_2} - \frac{v}{\rho_2}\right)$$

$$v(\xi_1,\xi_2, -\tfrac{c}{2}) = v_2 - \frac{t_2}{2}\left(\frac{1}{\alpha_2} w_{2,\xi_2} - \frac{v}{\rho_2}\right)$$

results in two compatibility conditions in terms of the components of the face displacements and the core shear functions $\gamma_1$ and $\gamma_2$ ; namely,

$$u_1\left(1 - \frac{c}{2\rho_1}\right) - u_2\left(1 + \frac{c}{2\rho_1}\right) + \frac{1}{\bar{\alpha}_1} w_{1,\xi_1}\left[\frac{c + t_1}{2} - \frac{c^2 + 3t_1c}{6\rho_1}\right]$$

$$+ \frac{1}{\bar{\alpha}_1} w_{2,\xi_1}\left[\frac{c + t_2}{2} + \frac{c^2 + 3t_2c}{6\rho_1}\right] - \gamma_1 c$$

$$+ \frac{c^3}{12F\bar{\alpha}_1}\left[\frac{G_1}{\bar{\alpha}_1\bar{\alpha}_2}(\bar{\alpha}_2\gamma_1)_{,\xi_1} + \frac{G_2}{\bar{\alpha}_2\bar{\alpha}_1}(\bar{\alpha}_1\gamma_2)_{,\xi_2}\right]_{,\xi_1} = 0 \qquad (105)$$

$$v_1\left(1-\frac{c}{2\rho_2}\right)-v_2\left(1+\frac{c}{2\rho_2}\right)+\frac{1}{\bar{\alpha}_2}w_{1,\xi_2}\left[\frac{c+t_1}{2}-\frac{c^2+3t_1c}{6\rho_2}\right]$$

$$+\frac{1}{\bar{\alpha}_2}w_{2,\xi_2}\left[\frac{c+t_2}{2}+\frac{c^2+3t_2c}{6\rho_2}\right]-\gamma_2c$$

$$+\frac{c^3}{12F\bar{\alpha}_2}\left[\frac{G_1}{\bar{\alpha}_1\bar{\alpha}_2}(\bar{\alpha}_2\gamma_1)_{,\xi_1}+\frac{G_2}{\bar{\alpha}_2\bar{\alpha}_1}(\bar{\alpha}_1\gamma_2)_{,\xi_2}\right]_{,\xi_2}=0 \qquad (106)$$

The boundary conditions (Equation (98)) allow elimination of the interface loads $q_1$, $q_2$, and $q_3$ from the Euler equations of the faces; then by use of Equations (100) and (103), resulting from integration of the equilibrium equations of the core, the only unknown functions remaining are the six face displacement components and the two core shear functions. Each of these eight functions depends on the surface coordinates $\xi_1$ and $\xi_2$ only.

In addition to these six equilibrium equations for the faces, the two compatibility conditions (Equations (105) and (106)) have to be satisfied to guarantee a unique solution. The last boundary condition for the core, Equation (99), also has to be satisfied in addition to the eight boundary conditions obtained for the two faceplates.

If it is assumed that the boundary load $q^*$ is distributed over the depth of the core in the same way as the internal shear stresses,

$$q^*(\zeta) = q^*(0)\,(1-2\zeta/\rho_n) \equiv \bar{q}^*(1-2\zeta/\rho_n)$$

then the ninth boundary condition can be written in final form as

$$G_1\gamma_1c_{21}+G_2\gamma_2c_{11}+\bar{q}^* = 0 \qquad \text{or}$$

$$\delta\left[w_1\left(1-\frac{c}{3\rho_s}\right)+w_2\left(1+\frac{c}{3\rho_s}\right)\right.$$
$$\left.+\frac{c^2}{6F}\left\{\frac{G_1}{\bar{\alpha}_1\bar{\alpha}_2}(\bar{\alpha}_2\gamma_1)_{,\xi_1}+\frac{G_2}{\bar{\alpha}_2\bar{\alpha}_1}(\bar{\alpha}_1\gamma_1)_{,\xi_2}\right\}\right]=0 \qquad (107)$$

According to Saint-Venant's principle, replacement of the actual boundary load

$$\int_{-c/2}^{c/2} q^*(\zeta)\, \alpha_s d\zeta$$

by the statically equivalent load $\bar{q}^* c\alpha_s$ will affect the stress distribution in a limited edge zone; it also affects the plane-stress components. However, as these stresses have been ignored in the present theory, Equation (107) is the appropriate boundary condition of the core. If the plane-stress components are important, then a more complete treatment of the core requires consideration of other than shear loads at the boundary.

## APPENDIX III

## BUCKLING EQUATIONS FOR AN AXIALLY COMPRESSED CIRCULAR CYLINDRICAL SANDWICH CONSTRUCTION

For an axially loaded cylinder, the shallow-shell equations can be used with great confidence to determine buckling or postbuckling behavior. The following substitutions are made in order to specialize Equations (19) and (20) for this case (see Figure 3). For the reference surface,

$$\xi_1 = x \qquad \bar{\alpha}_1 = 1 \qquad \rho_1 = \infty = \rho_n$$

$$\xi_2 = \varphi \qquad \bar{\alpha}_2 = R = \rho_2 = \rho_s$$

For face 1 (outer face):

$$\alpha_1 = 1 \qquad \rho_1 = \infty$$

$$\alpha_2 = R_1 = \rho_2 \tag{108}$$

For face 2 (inner face):

$$\alpha_1 = 1 \qquad \rho_1 = \infty$$

$$\alpha_2 = R_2 = \rho_2$$

At the boundary, $x = 0$ :

$$\xi_s = -\varphi \qquad \alpha_s = R_i = \rho_s \qquad d_s = -v \qquad N_s = -N_\varphi \qquad M_s = -M_\varphi$$

$$\xi_n = x \qquad \alpha_n = 1 \quad \rho_n = \infty \qquad d_n = u \qquad N_n = N_x \qquad M_n = M_x$$

$$c_{11} = 0 = c_{22} \qquad \tau = 0 \qquad N_z = N_\zeta$$

$$c_{12} = 1 = - c_{21}$$

$$d_{11} = 0 = d_{22}$$

$$d_{12} = - 1 = - d_{21} \tag{109}$$

At the boundary, $x = L$ :

$$\xi_s = \varphi \qquad \alpha_s = R_1 = \rho_s \qquad d_s = v \qquad N_s = N_\varphi \qquad M_s = M_\varphi$$

$$\xi_n = x \qquad \alpha_n = 1 \quad \rho_n = \infty \qquad d_n = - u \qquad N_n = - N_x \qquad M_n = - M_x$$

$$c_{11} = 0 = c_{22} \qquad \tau = 0 \qquad N_z = - N_\zeta$$

$$c_{12} = -1 = -c_{21}$$

$$d_{11} = 0 = d_{22}$$

$$d_{12} = 1 = - d_{21} \tag{110}$$

Compatibility conditions:

$$u_1 - u_2 - c\gamma_1 + \frac{c^3}{12F}\left[G_1\gamma_{1,x} + G_2\gamma_{2,\bar{\varphi}}\right]_{,x}$$

$$+ \frac{c + t_1}{2} w_{1,x} + \frac{c + t_2}{2} w_{2,x} = 0 \tag{111a}$$

$$v_1\left(1 - \frac{c}{2R}\right) - v_2\left(1 + \frac{c}{2R}\right) - c\gamma_2 + \frac{c^3}{12F}\left[G_1\gamma_{1,x} + G_2\gamma_{2,\bar{\varphi}}\right]_{,\bar{\varphi}}$$

$$+\left(\frac{c + t_1}{2} - \frac{c^2 + 3t_1c}{6R}\right) w_{1,\bar{\varphi}}$$

$$+\left(\frac{c + t_2}{2} + \frac{c^2 + 3t_2c}{6R}\right) w_{2,\bar{\varphi}} = 0 \qquad (111b)$$

Equilibrium equations for faces 1 and 2 :

$$C_i(\epsilon_1 + \nu\epsilon_2)_{,x} + C_i\frac{1-\nu}{2}\gamma_{12,\varphi} \mp G_1\gamma_1\left(1 \mp \frac{c}{2R}\right) = 0 \qquad (111c)$$

$$C_i(\epsilon_2 + \nu\epsilon_1)_{,\varphi} + C_i\frac{1-\nu}{2}\gamma_{12,x} \mp G_2\gamma_2\left(1 \mp \frac{c}{R}\right) = 0 \qquad (111d)$$

$$\left[D_i(\kappa_1 + \nu\kappa_2)_{,x} + D_i(1-\nu)\,\kappa_{12,\varphi}\right]_{,x} + \left[D_i(\kappa_2 + \nu\kappa_1)_{,\varphi} + D_i(1-\nu)\,\kappa_{12,x}\right]_{,\varphi}$$

$$- C_i(\epsilon_1 + \nu\epsilon_2\,\kappa_1 - C_i(\epsilon_2 + \nu\epsilon_1)\left(\kappa_2 - \frac{1}{R_i}\right) - 2C_i\frac{1-\nu}{2}\gamma_{12}\kappa_{12}$$

$$\pm F\left(1 \mp \frac{c}{2R}\right)\left[\frac{w_1 - w_2}{c} \mp \frac{c}{2}\,\frac{G_1}{F}\gamma_{1,x} \mp \frac{c}{2}\left(1 \mp \frac{c}{3R}\right)\frac{G_2}{F}\gamma_{2,\bar{\varphi}}\right]$$

$$-\frac{t_i}{2}\left[\left(1 \mp \frac{c}{2R}\right)G_1\gamma_{1,x} + \left(1 \mp \frac{c}{R}\right)G_2\gamma_{2,\bar{\varphi}}\right] = 0 \qquad (111e)$$

Boundary conditions at $x = 0,L$ :

$$C_i(\epsilon_1 + \nu\epsilon_2) - N_{xi} = 0 \quad \text{or} \quad \delta u_i = 0 \qquad (112a)$$

$$C_i \frac{1-\nu}{2} \gamma_{12} - N_{\varphi i} = 0 \quad \text{or} \quad \delta v_i = 0 \tag{112b}$$

$$- C_i(\epsilon_1 + \nu\epsilon_2)\, w_{,x} - C_i \frac{1-\nu}{2} \gamma_{12} w_{,\varphi} - \frac{t_i}{2} G_1\gamma_1 \left(1 \mp \frac{c}{2R}\right) + N_\zeta - M_{\varphi i,\varphi}$$

$$- D_i(\kappa_1 + \nu\kappa_2)_{,x} - 2D_i(1-\nu)\, \kappa_{12,\varphi} = 0 \quad \text{or} \quad \delta w_i = 0 \tag{112c}$$

$$- D_i(\kappa_1 + \nu\kappa_2) - M_{\varphi i} = 0 \quad \text{or} \quad \delta w_i \tag{112d}$$

$$- cG_1\gamma_1 + \bar{q}^* = 0 \quad \text{or}$$

$$\delta \left[ w_1 \left(1 - \frac{c}{3R}\right) + w_2 \left(1 + \frac{c}{3R}\right) + \frac{c^2}{6F} \left\{ G_1\gamma_{1,x} + G_2\gamma_{2,\bar{\varphi}} \right\} \right] = 0 \tag{112e}$$

Under the shallow-shell assumptions, the strains and changes of the curvature are simply

$$\left.\begin{aligned}
\epsilon_1 &= \epsilon_x \doteq u_{,x} + \frac{1}{2} w_{,x}^2 \\
\epsilon_2 &= \epsilon_\varphi = v_{,\varphi} + \frac{w}{R} + \frac{1}{2} w_{,\varphi}^2 \\
\gamma_{12} &= \gamma_{x\varphi} = u_{,\varphi} + v_{,x} + w_{,x}w_{,\varphi} \\
\kappa_1 &= \kappa_x = w_{,xx} \\
\kappa_2 &= \kappa_\varphi = w_{,\varphi\varphi} \\
\kappa_{12} &= \kappa_{x\varphi} = w_{,x\varphi}
\end{aligned}\right\} \tag{113}$$

If it is assumed that the cylinder deforms uniformly under a uniformly applied axial compressive load up to a critical value of the load (that is, no edge loads are present other than the compressive force $N_{xi}$ ), then the subcritical state of deformation is characterized by the uniform lateral displacements $w_1^*$ and $w_2^*$ , the absence of shear deformations $\gamma_1$ and $\gamma_2$ of the core, and the circumferential displacements $v_1$ and $v_2$ . Also, in the present notation, the changes of curvature as well as derivatives in the $\varphi$-direction are zero.

From (111a) then, it follows that $u_1 = u_2 = u^*$ , Equations (111b,d) and the boundary conditions (112b,c,d,e) are satisfied identically. Equations (111c) and the boundary conditions (112a) show that

$$N_{xi} = C_i(\epsilon_1 + \nu\epsilon_2) = C_i\left(u^*_{,x} + \nu \frac{w_i^*}{R_i}\right) \tag{114}$$

while Equation (111e) becomes

$$\frac{C_1}{R_1}\left[\frac{w_1^*}{R_1} + \nu u^*_{,x}\right] + F\left(1 - \frac{c}{2R}\right)\frac{w_1^* - w_2^*}{c} = 0$$

$$\frac{C_2}{R_2}\left[\frac{w_2^*}{R_2} + \nu u^*_{,x}\right] - F\left(1 + \frac{c}{2R}\right)\frac{w_1^* - w_2^*}{c} = 0 \tag{115}$$

Subcritical deformations are marked by an asterisk.

Multiplication of the first of Equation (115) by $R_1$ and of the second by $R_2$ shows, after addition of the two results, that the total circumferential force vanishes; that is,

$$C_1\left(\frac{w_1^*}{R_1} + \nu u^*_{,x}\right) + C_2\left(\frac{w_2^*}{R_2} + \nu u^*_{,x}\right) = 0$$

Finally, solution of Equations (114) and (115) yields

$$
\begin{aligned}
N_{x1} &= u^*_{,x}\, C_1 \left[ (1-\nu^2) \left\{ 1 + \frac{c/R}{1 + \dfrac{C_2}{C_1}(1 - c/2R) + \dfrac{C_2 c}{FR^2}(1 + c/2R)} \right\} \right] \\
N_{x2} &= u^*_{,x}\, C_2 \left[ (1-\nu^2) \left\{ 1 - \frac{c/R}{1 + \dfrac{C_1}{C_2}(1 + c/2R) + \dfrac{C_1 c}{FR^2}(1 - c/2R)} \right\} \right]
\end{aligned}
\qquad (116)
$$

This determines the ratio of $N_{x1}$ and $N_{x2}$ necessary to achieve a uniform prebuckling deformation.

For a thin shell $c/R \ll 1$ and, in this case, neglecting terms of the order $c/R$ gives

$$N_{xi} = u^*_{,x}\, C_i(1-\nu^2) \qquad (117)$$

Poisson's ratio has been taken to be the same for both faces. For simplicity, attention may be restricted to the case of completely similar facings, $(Et)_1 = (Et)_2$ . Then, simply,

$$N_{x1} = N_{x2} = - N_x \qquad (118)$$

If the prebuckling deformations are subtracted from the total deformations and the buckling deformations so obtained are substituted into the strain and curvature expressions (Equation (113)), Equations (111) and (112) describe the postbuckling behavior of the thin sandwich shell under the assumptions made.

For a determination of the critical load, products of buckling strains and/or stresses can be omitted and the "buckling equations" for a thin sandwich cylinder with similar facings are obtained as:

$$C\left[u_{1,x} + \nu\left(v_{1,\varphi} + \frac{w_1}{R}\right)\right]_{,x} + C\,\frac{(1-\nu)}{2}\left[u_{1,\varphi} + v_{1,x}\right]_{,\varphi} - G_1\gamma_1 = 0 \quad (119a)$$

$$C\left[u_{2,x} + \nu\left(v_{2,\varphi} + \frac{w_2}{R}\right)\right]_{,x} + C\,\frac{(1-\nu)}{2}\left[u_{2,\varphi} + v_{2,x}\right]_{,\varphi} + G_1\gamma_1 = 0 \quad (119b)$$

$$C\left[v_{1,\varphi} + \frac{w_1}{R} + \nu u_{1,x}\right]_{,\varphi} + C\,\frac{(1-\nu)}{2}\left[u_{1,\varphi} + v_{1,x}\right]_{,x} - G_2\gamma_2 = 0 \quad (119c)$$

$$C\left[v_{2,\varphi} + \frac{w_2}{R} + \nu u_{2,x}\right]_{,\varphi} + C\,\frac{(1-\nu)}{2}\left[u_{2,\varphi} + v_{2,x}\right]_{,x} + G_2\gamma_2 = 0 \quad (119d)$$

$$D\nabla^4 w_1 - \frac{t}{2}\left[G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right] + F\left[\frac{w_1 - w_2}{c} - \frac{c}{2F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right)\right]$$

$$+ \frac{c}{R}\left[v_{1,\varphi} + \frac{w_1}{R} + \nu u_{1,x}\right] + N_x w_{1,xx} = 0 \quad (119e)$$

$$D\nabla^4 w_2 - \frac{t}{2}\left[G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right] - F\left[\frac{w_1 - w_2}{c} + \frac{c}{2F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right)\right]$$

$$+ \frac{c}{R}\left[v_{2,\varphi} + \frac{w_2}{R} + \nu u_{2,x}\right] - N_x w_{2,xx} = 0 \quad (119f)$$

$$u_1 - u_2 + \frac{t + c}{2}\left(w_1 + w_2\right)_{,x} - \gamma_1 c + \frac{c^3}{12F}\left(G_1\gamma_{1,x} + G_2\gamma_{2,\varphi}\right)_{,x} = 0 \quad (119g)$$

$$v_1 - v_2 + \frac{t + c}{2}\left(w_1 + w_2\right)_{,\varphi} - \gamma_2 c + \frac{c^3}{12F}\left(G_1\gamma_{1,x} - G_2\gamma_{2,\varphi}\right)_{,\varphi} = 0 \quad (119h)$$

Appropriate boundary conditions for the problem under consideration are to be chosen from the following set of pairs:

At $x = 0, L$ :

$$C\left[u_{1,x} + \nu(v_{1,\varphi} + w_1/R_1)\right] = 0 \quad \text{or} \quad u_1 = 0 \tag{120a}$$

$$C\left[u_{2,x} + \nu(v_{2,\varphi} + w_2/R_2)\right] = 0 \quad \text{or} \quad u_2 = 0 \tag{120b}$$

$$C\,\frac{(1-\nu)}{2}\left[u_{1,\varphi} + v_{1,x}\right] = 0 \quad \text{or} \quad v_1 = 0 \tag{120c}$$

$$C\,\frac{(1-\nu)}{2}\left[u_{2,\varphi} + v_{2,x}\right] = 0 \quad \text{or} \quad v_2 = 0 \tag{120d}$$

$$D\left[w_{1,xxx} + (2-\nu)\,w_{1,\varphi\varphi x}\right] - \frac{t}{2}\,G_1\gamma_1 - N_x w_{1,x} = 0 \quad \text{or} \quad w_1 = 0 \tag{120e}$$

$$D\left[w_{2,xxx} + (2-\nu)\,w_{2,\varphi\varphi x}\right] - \frac{t}{2}\,G_1\gamma_1 - N_x w_{2,x} = 0 \quad \text{or} \quad w_2 = 0 \tag{120f}$$

$$D\left[w_{1,xx} + \nu w_{1,\varphi\varphi}\right] = 0 \quad \text{or} \quad w_{1,x} = 0 \tag{120g}$$

$$D\left[w_{2,xx} + \nu w_{2,\varphi\varphi}\right] = 0 \quad \text{or} \quad w_{2,x} = 0 \tag{120h}$$

$$CG_1\gamma_1 = 0 \quad \text{or} \quad w_1 + w_2 + \frac{c^2}{6F}\,(G_1\gamma_{1,x} + G_1\gamma_{1,\varphi}) = 0 \tag{120i}$$

## APPENDIX IV

## EXPRESSIONS FOR CRITICAL LOADS IN THE PLASTIC RANGE

The loss of stability of a sandwich cylinder under uniform axial compression may occur at stress levels beyond the elastic limit. In this case, the change in strain energy due to a small perturbation of the prestressed state has to be determined, taking into account the sign of the changes in strain. Stress increments and decrements are related to changes in strains by different laws. This results in modified expressions for the bending stiffness and extensional stiffness of the faceplates and presumably also for the core stiffness.

For the behavior of the faceplates under a prestress beyond the elastic limit, the following generalization of Hooke's law is used:

$$\delta\sigma_x = \frac{E^*}{1-\mu^{*2}} (\delta\epsilon_x + \mu^*\delta\epsilon_\varphi)$$

$$\delta\sigma_\varphi = \frac{E^*}{1-\mu^{*2}} (\delta\epsilon_\varphi + \mu^*\delta\epsilon_x) \tag{121}$$

where $E^* = E$ or $E_t$ and $\mu^* = \nu$ or $\mu$ according as a stress intensity decreases or increases. $E$ and $E_t$ are Young's modulus and the tangent modulus as obtained from a uniaxial stress-strain curve (possibly in a tensile test), and $\nu$ and $\mu$ are Poisson's ratios for the elastic and the inelastic range; plastic deformations are assumed to be volume preserving $(\mu = 1/2)$. The stress intensity $\bar{\sigma}$ is defined here as the square root of the second invariant of the stress deviator tensor $J_2$. In the case of the usual plane stress approximation for face sheets,

$$\bar{\sigma} = \sqrt{J_2} = \frac{1}{\sqrt{3}} (\sigma_x^2 + \sigma_\varphi^2 - \sigma_x\sigma_\varphi + 3\tau_{x\varphi}^2)^{1/2}$$

so

$$\delta\bar{\sigma} = \frac{1}{6\bar{\sigma}} \left[ (\sigma_x - \sigma_\varphi)\,(\delta\sigma_x - \delta\sigma_\varphi) + 6\tau_{x\varphi}\delta\tau_{x\varphi} \right]$$

For a perfect cylinder in the prebuckled state, $\sigma_\varphi = \tau_{x\varphi} = 0$ . Hence,

$$\delta\bar{\sigma} = \frac{\sigma_x(\delta\sigma_x - \delta\sigma_\varphi)}{2\,|\sigma_x|\,\sqrt{3}} \tag{122}$$

SYMMETRIC WRINKLING MODE

For this mode, $w_1 = -w_2$ , while the other displacement components and the core shear angle are essentially zero. The only perturbations of the prestrained state are a bending strain component $\delta\epsilon_x = -zw,_{xx}$ and a circumferential strain component $\delta\epsilon_\varphi = w/R$ . Then,

$$\delta\bar{\sigma} = -\frac{E^*}{2\,(1+\mu^*)\,\sqrt{3}}\,(\delta\epsilon_x - \delta\epsilon_\varphi)$$

But

$$|\delta\epsilon_x| \,/\, |\delta\epsilon_\varphi| = \left| \frac{2z}{t} \cdot \frac{Rt\pi^2}{2\lambda^2} \right| \approx \left| \frac{2z}{t} \right| \beta^2/\sqrt{\zeta} \gg 1$$

except near the middle surface $(z = 0)$ , so the sign of $\delta\bar{\sigma}$ is the opposite of the sign of $\delta\epsilon_x$ and the face bending stiffness is taken as

$$D^* \simeq \bar{E}^*\,\frac{t^3}{12} \tag{123}$$

where the reduced modulus $\bar{E}^*$ is the well known double modulus:

$$\bar{E}^* = 4\left(\frac{E}{1-\nu^2} + \frac{E_t}{1-\mu^2}\right)\left(\sqrt{\frac{E}{1-\nu^2}} + \sqrt{\frac{E_t}{1-\mu^2}}\right)^{-2} = \frac{4\bar{E}\,\bar{E}_t}{(\sqrt{\bar{E}} + \sqrt{\bar{E}_t})^2}$$

According to Equation (40), the wrinkling load is determined by

$$k = \frac{\sigma_x t c^2}{\pi^2 D^*} \approx \beta^2 + \varphi^*/\beta^2 = \beta^2 + \frac{2F^* c^3}{\pi^4 D^* \beta^2} \tag{124}$$

In this expression, $F^*$ is a reduced core modulus, As the core is assumed to be stress-free up to the point of instability, $F^*$ should be equal to the elastic value $F$ . Actually, the core is attached to the faces, and the outer core layers will undergo the same inplane strains as face layers that are strained beyond their elastic limit. In addition, buckling of cell walls may be a source of deterioration of the core stiffness.

An upper limit for the wrinkling stress is found by taking $F^* = F$ ; a low value results if it is assumed that $F^* = F \cdot \bar{E}_t / \bar{E}$ . In both cases, both $D^*$ and $F^*$ are completely determined by the face stress $\sigma_x$ and will be stationary if $\sigma_x$ reaches an extreme value. Hence, a critical situation prevails, similar to the elastic case, if $\beta = (\varphi^*)^{1/4}$ ; the corresponding critical load parameter is

$$\left(k_{cr}\right)_{wr} = 2\sqrt{\varphi^*}$$

On the basis of the two extreme assumptions regarding the magnitude of $F^*$ , the critical stress is assumed to be bounded by

$$\frac{2\sqrt{\frac{2t}{3c} F \bar{E}_t^2}}{\sqrt{\bar{E}} + \sqrt{\bar{E}_t}} < \sigma_x < \frac{2\sqrt{\frac{2t}{3c} F \bar{E}\bar{E}_t}}{\sqrt{\bar{E}} + \sqrt{\bar{E}_t}} \tag{125}$$

CLASSICAL MODE

The classical mode of instability is characterized by a negligible transverse strain and a wavelength that is several times the core thickness. The ratio of bending strains and neutral surface strains is very small for the face layers.

The essential changes in the state of strain are

$$\delta\epsilon_x = u,_x = - U\,\pi/\lambda \sin \pi\cdot x/\lambda \qquad \text{and}$$

$$\delta\epsilon_\varphi = w/R = W/R \sin \pi\cdot x/\lambda$$

Hence,

$$\delta\bar{\sigma}_i = \frac{E_i^*}{2\,(1+\mu_i^*)\sqrt{3}} \cdot \frac{U_i\pi}{\lambda} \cdot \left[1 + \frac{\lambda}{\pi R}\,\frac{W_i}{U_i}\right] \sin\frac{\pi x}{\lambda}$$

Now as $W_1 \approx W_2$, but $U_1 \approx - U_2$, the term in front of the square bracket and the second term between the brackets have a different sign for the two face layers. Then the sign of $\delta\epsilon_{eff}$ will also differ for the two faces if

$$\frac{\lambda}{\pi R}\,\frac{W_i}{U_i} < 1$$

This condition is met in most practical cases, and then $\delta\bar{\sigma}$ will have opposite signs in the two face layers. The buckling criterion for the classical mode, according to Equation (44), can be written as

$$N_x = \frac{\dfrac{D_s^*}{2}\,\dfrac{\pi^2}{\lambda^2}}{1 + \dfrac{D_s^*}{G_1^*(c+t)^2/c}\,\dfrac{\pi^2}{\lambda^2}} + \left[\left(\frac{E^*t}{R^2}\right)_1 + \left(\frac{E^*t}{R^2}\right)_2\right]\frac{\lambda^2}{2\pi^2} \qquad (126)$$

and the appropriate expressions for the reduced bending and hoop stiffnesses of the sandwich shell are

$$D_s^* = 2\,\frac{\bar{E}\,\bar{E}_t}{\bar{E} + \bar{E}_t}$$

and

$$\left[\frac{E^* t}{R^2}\right]_1 + \left[\frac{E^* t}{R^2}\right]_2 = \frac{E + E_t}{2} \frac{t}{R^2}$$

If two bounds for the core shear stiffness are chosen as

$$G_1 \frac{\bar{E}_t}{E} < G_1^* < G_1$$

then all stiffness values are stationary whenever the value of the load $N_x$ is stationary and bounds for the classical buckling stress are obtained as

$$\frac{c + t}{R} \sqrt{\frac{E + E_t}{\bar{E} + \bar{E}_t} \bar{E} \bar{E}_t} - \frac{ct}{2R^2} \frac{E + E_t}{\bar{E} + \bar{E}_t} \cdot \frac{\bar{E}^2}{G_1} < \sigma_x$$

$$< \frac{c + t}{R} \sqrt{\frac{E + E_t}{\bar{E} + \bar{E}_t} \bar{E} \bar{E}_t} - \frac{ct}{2R^2} \frac{E + E_t}{\bar{E} + \bar{E}_t} \frac{\bar{E} \bar{E}_t}{G_1} \tag{127}$$

ANTISYMMETRIC WRINKLING

Although, in general, the critical stress for elastic antisymmetric wrinkling is shown to be larger than the elastic symmetric wrinkling stress, the antisymmetric result depends on both $F$ and $G$, while the symmetric result is independent of $G$. Hence, if both quantities are affected by plastic deformations, the ratio of symmetric and antisymmetric wrinkling stresses will change. In the antisymmetric-mode, face bending strains are usually larger than face neutral surface strains. If, again, upper and lower bounds for the core stiffness quantities are assumed in the form

$$\frac{\bar{E}_t}{\bar{E}} (F, G) < F^*, G^* < F, G$$

then all stiffness quantities assume stationary values at the critical value of the stress. Equation (49) can be written in the form

$$N_x = 2\,\frac{c+t}{t}\sqrt{\frac{6F^*D^*}{c^3} - \frac{12}{c^2t}\,\frac{F^*D^*}{G_1^*}} \tag{128}$$

and the antisymmetric wrinkling stress is assumed to be bounded by

$$\frac{2(c+t)}{c}\,\frac{\sqrt{\frac{2t}{c}F\bar{E}_t^2}}{\sqrt{\bar{E}}+\sqrt{\bar{E}_t}} - \frac{4t^2}{c^2}\,\frac{F}{G_1}\,\frac{\bar{E}\,\bar{E}_t}{\left[\sqrt{\bar{E}}+\sqrt{\bar{E}_t}\right]^2} < \sigma_x$$

$$< \frac{2(c+t)}{c}\,\frac{\sqrt{\frac{2t}{c}F\bar{E}\,\bar{E}_t}}{\sqrt{\bar{E}}+\sqrt{\bar{E}_t}} - 4\,\frac{t^2}{c^2}\,\frac{F}{G_1}\,\frac{\bar{E}\,\bar{E}_t}{\left[\sqrt{\bar{E}}+\sqrt{\bar{E}_t}\right]^2} \tag{129}$$

It must be noted that the prevalence of face bending strain over face stretching is less pronounced in the antisymmetric wrinkling mode than it is in the general instability mode. If the two types of strain became of comparable magnitude, the change in the stress intensity is a more complicated function of the coordinates and a more complete study of the inelastic behavior is mandatory.